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THEORY OF SHEAVES

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR
THE DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Theory of Sheaves", submitted by Paul L. Manley, B. Sc., in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

The concept of a sheaf unifies and axiomatizes what is shared in common by associating with each point of a topological space (base space) an algebraic structure (stalk) and then treating the totality of these structures as a topological space (sheaf space). In practice the stalks are not chosen arbitrarily but are defined in some specific way in terms of the properties of the base space, expressing in this way information about the local property of the base space and the corresponding set of all sections giving the global property. The concept of a sheaf gives a workable mathematical formulation for the intuitive notion of variation of a structure.

Elementary aspects of sheaf theory are developed in detail. In the first chapter the sheaf, presheaf and section are defined and the relation of (associated) sheaf and (canonical) presheaf is established. Next algebraic concepts (homomorphisms of sheaves and presheaves, sub and quotient sheaves and presheaves, exact sequences) and the operations (direct sum, tensor product) are developed. In chapter two the coherent theory of sheaves is developed while in the last chapter the algebraic and analytic theories of sheaves are briefly described.

INTRODUCTION

Modes come and go in mathematics as in other fields of human endeavor. Often new modes have difficulty breaking through until they are greeted with acclaim, copied and multiplied by the debutants, and then only to become old-fashioned and disdained by the still younger set. In the mean-time the old modes show great reluctance to disappear for there are always some faithful souls who prefer the Paris of twenties to, say, the Princeton mode of the fifties. It is never safe to opine that a particular field in mathematics is dead or has outlived its usefulness. Pronouncements to that effect have often been made perhaps with malice toward some, but usually have been belied by later events.

Although mathematicians affirm the universality of their subject, they deny that it possesses any absolute qualities of truth. In fact, one of their favorite descriptions of mathematics is Bertrand Russell's witty summation: "the subject in which we never know what we are talking about nor whether what we are saying is true". This description reflects not modesty but as proud and Promethean a boast as man has ever made. In effect mathematicians are saying that their work can apply to any part of our world and universe that might be constructed along logical lines. They are saying that mathematics reaches into a realm of such ultimate sophistication that the truth or non-truth of any given premise no longer matters. What does matter, they say, is that the premise be

correctly reasoned to its conclusion. Using this criterion, a mathematician could unblinkingly assume that the moon was made of green cheese, and convincingly argue, through a series of further premises, to the conclusion that astronauts should carry crackers.

Mathematics is not so much a body of knowledge as a special kind of language, distinguished by its ability for clarity and particularly well suited to develop logical arguments. The grammar of the language - its proper usage - is determined by the rules of logic, and its vocabulary consists of symbols representing assorted concepts. The power of mathematics is no more and no less than the power of pure reason.

Over the years, mathematicians have tried to improve their knowledge of their subject, and with each advance there have arisen new questions concerning the foundations upon which mathematics is built. It was not until some of the intuitively obvious facts were shown to be false that mathematicians began to question their assumptions. In order to question the assumptions, it was often necessary to state an intuitive notion quite precisely. When this was done, it soon became obvious that a better understanding of mathematics demanded a language that was as precise as man could make it.

A large portion of the so dubbed "modern mathematics has been concerned with this improvement of the language. The need for a precise language has given rise to "precise" mathematics, and not the misnomer "modern" mathematics.

In both range and remoteness, modern mathematics defies easy description. In general, however, it has developed along two lines: on the one hand success and conquest - the ability to solve problems; on the other hand soul-searching and contemplation - an uncertainty as to the nature and aim of ultimate mathematical abstraction. Both these strands are woven into the many-splendored fabric of mathematics today.

On the contemplative side, two of the most notable developments are set theory and symbolic logic. Set theory, among other things, provides a technique for "anatomizing the infinite"; whereas symbolic logic attempts to reduce all human reasoning to a mathematical notation. Symbolic logic has produced one of the most curious and influential theorems in all modern mathematics - Godel's theorem, which shows that no useful branch of mathematics can be constructed on a consistent set of axioms without raising questions unanswerable within the framework of the axioms themselves. For mathematicians as a whole, the implication is that the subject will never be complete - that there is limitless scope for fishing in the mathematical ocean.

Both set theory and symbolic logic are abetted by a third form of mathematics, structure theory, which accounts for the other mathematicians - of course, with the exception of the dispensable appendage of the "numericals".

The nature, trend, feature and aim of modern mathematics

may be stated in the following one-sentence remarks. It is the essence of mathematics that it concerns itself with those relations which lie so deep in the nature of things that they occur in the most varied situations. The tendency is to isolate almost any set of properties from its original context, to name and to develop an abstract theory. A characteristic feature is its emphasis on the study of objects defined partly algebraically and partly topologically, or partly algebraically and partly differentiably. The aim is to reveal the structure that is inherent in mathematics; that is to say, "we have grasped it mathematically only if we know the structure".

The algebraic-topological methods have in many cases been suggested by the respective fields, and they have, conversely, strongly influenced them and contributed to their recent development. For instance, the algebraization of mathematics has forced modules into a central position, tensor products arose from differential geometry, categories and homological algebra from topology.

During the period 1935-40, the subject of topology produced two new tools, first the homotopy groups of Hurewicz and secondly the concept of a fiber space as developed by Whitney, Stiefel and Hopf; during 1940-45, homology and cohomology as developed Hopf, Eckmann, Eilenberg, MacLane, Hochschild, Chevalley and others; and during 1945-50, the notion of a sheaf introduced by Leray in order to study homological properties of a continuous map.

The fundamental problem and last problem of classical

topology consists in determining the necessary and sufficient conditions for two given topological spaces to be homeomorphic. The direct and more or less intuitive methods based on the knowledge related to simple examples of topological spaces have long since been abandoned. Recent investigation is attempting to arrive at the necessary conditions by associating to a topological space an algebraic structure that depends on the topological space only, that is, without the homeomorphisms. Consequently the problem of the classification of the homeomorphisms of the topological space can be at least reduced partly to a problem of classification of isomorphism of algebraic structures for which the recent algebra, homological algebra, offers much information and many ways of approach. Actually, homological algebra was invented as part of the process of formalizing algebraic topology.

The algebraic structure associated to a topological space is the homology group (or cohomology group) which deals with the formal properties of maps and their composites. More intuitively, homology provides an algebraic picture of topological spaces, assigning to each space X a family of abelian groups $H_0(X), \dots, H_n(X), \dots$, and to each continuous map $f : X \rightarrow Y$ a family group homomorphisms $f_n : H_n(X) \rightarrow H_n(Y)$. Properties of the space or the map can often be found from properties of the group H_n or the homomorphism f_n . Of course, the efficacy of an algebraic technique depends not only on the richness of the algebraic structure but also on the faithfulness wherewith the

topological property is reproduced in the algebraic model. A particular algebraic technique that is used is a functor F passing from a topological category C to an algebraic category C' and the question is then asked as to what categorical features of C and C' are preserved by F .

It is a fundamental principle in the investigation of any system that one needs to know not only its internal structure but also its representations. So, in addition to determining all the homology groups of a space, one also determines all their representations, giving rise in this way to the usual pattern of chains, cycles and boundaries.

Two general points of view in various branches of mathematics is the formulations of invariants and covariants in abstract spaces, and the characterization of mathematical entities by extremum properties. The invariants in algebraic topology, such as, homotopy groups, homology groups and cohomology operations, are all homotopic in nature. Knowledge about the so-called "pure" and yet non-homotopic topological invariants is still scant and fragmentary.

There has been in recent years trend to unification which clarifies the common ideas and is helpful for applications. At its origin is on one hand, the categorical formulation due to Eilenberg and MacLane and the axiomatic approach to homology due to Eilenberg and Steenrod, and, on the other hand, the successful efforts made in homotopy, fiber bundle and sheaf theory, etc. It appears more

and more clear that there is a common framework for the various concepts and methods, simple at least in its basic ideas.

One of the chapters of the vast field of algebraic topology is that the sheaves. A sheaf is just one of the examples in the recent trend toward hybridization of mathematical structures. It is, in particular, an algebraic-differential-topological structure from which are developed the three specialized theories: algebraic sheaves, analytic sheaves and differential sheaves.

The concept of a sheaf like all meaningful mathematical concepts was maturing for some time in the body of other mathematical theories before it came to light, acquired a name and became the subject of an independent study. Essentially the concept of a sheaf unifies and axiomatizes what is shared in common by associating any set with or without structure to each point of a topological space. In this way, a sheaf may be regarded as a system of local coefficients over a topological space.

The concept of a sheaf gives a workable mathematical formulation for the intuitive concept of "variation" of a structure. A sheaf may be interpreted as describing some local property of the space over which it is defined and the set of all sections of the sheaf giving the corresponding global property of the space. More precisely, the cohomology modules of a space with a coefficient sheaf furnish the algebraic tool to formulate globally the properties of its local structure, where, in particular, the zero-dimensional cohomology

module with a coefficient sheaf may be interpreted as the module of all sections of the sheaf.

With the permeating influence of homological algebra and category theory on mathematics as a whole, sheaves may actually be considered as a chapter in the theory of the functor categories.

The theory of sheaves has been developing from three points of view: that of Cartan with his fine sheaves, that of Grothendieck with his injective sheaves and that of Godement with his flabby sheaves. Recent studies appear to indicate that the central problem in sheaf theory is that of extending sections of a sheaf, so that Godement's development is actually most appropriate.

As for applications, the development of the theory of sheaves, and its adaptation for use in the theory of functions of several complex variables (analytic and differential sheaves) has provided a new tool for dealing with integrals of differential forms. In abstract algebraic geometry, sheaves (algebraic sheaves) are fundamental. Moreover, they are used in proofs of several important isomorphism theorems in homology and in cohomology theories.

Although the study of sheaves is a prerequisite for the study of many newer fields of mathematics, in this exposition it is presented as a subject of interest in itself. The three main parts of the general theory of sheaves are: (1) Sheaves (algebraic, analytic, differential), maps, sections, operations on and constructions of sheaves, coherence and exact sequences; (2) Fine, injective and

flabby sheaves, methods of homological algebra in sheaf theory (injective resolutions, derived functors, spectral sequences, etc.); (3) Cohomology theory of sheaves.

Here only the rudiments of sheaf theory are developed, namely topics in (1) and the most elementary aspects of (3) . Many of the proofs stated are in various stages of incompleteness and some actually contain only suggestions for a possible proof. The following notation is used to indicate the sources from which the results and proofs were obtained. (1, (-)) indicates that the result without proof is stated in (1) of the bibliography and that the author supplied a proof or made suggestions for its proof. (2, (+)) indicates that the result with proof is stated in (2) . (3, (0)) indicates that the result without proof is stated in (3) and the proof was obtained from another source or from a variety of sources, some of which are not listed in the bibliography.

In any introductory study of a topic (such as this one) one danger is to avoid the temptation to include too many concepts and so to overwhelm the exposition with a host of definitions unsupported by a substantial body of results to indicate their power and relevance. This would lead to a poor mathematical significance ratio of number of theorems to number of definitions, and would convey the impression that the topic (sheaf theory) was an organism with a highly developed bone structure but little meat to clothe it. This work is indeed obsessed with just such a guilt!

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Chapter 1Sheaves over a Topological Space1. Definition of Sheaves

The definition of a sheaf below is historically the original one of topological nature; that is, sheaves are defined as particular topological spaces. This interpretation is more appropriate than the more recent algebraic one in the elementary theory of sheaves.

Sheaves were introduced by Leray (70), and the modified definition now used was given by Lazard (11).

Definition 1.1: A sheaf $\mathcal{S} = (S, \sigma, X)$ of (additive) abelian groups over a topological space X (base space) is a set S (sheaf space) provided with an onto map (projection map)

$$\sigma : S \longrightarrow X$$

and provided with two structures:

1. Algebraic structure

S_1 : For each $x \in X$ the set $\sigma^{-1}(x) = S_x$ (stalk at x) is an (additive) abelian group.

2. Topological structure

The sheaf space S is a topological space such that:

S_2 : σ is a local homeomorphism of S onto X .

S_3 : The composition law (addition) of the stalks is continuous in the topology of S .

Remark 1.1: Let $S \times S$ be the cartesian product of the sheaf space S , and let $S + S$ be the subspace consisting of those pairs (s_1, s_2) for which

$$\sigma(s_1) = \sigma(s_2) .$$

Addition is continuous means that

$$f : S + S \longrightarrow S$$

defined by

$$f(s_1, s_2) = s_1 + s_2$$

is continuous. In other words, if $s, s' \in S$ and $\sigma(s) = \sigma(s')$ then given an open set G containing $s + s'$, there exist open sets H, K with $s \in H, s' \in K$ such that if $s_1 \in H, s'_2 \in K$ and $\sigma(s_1) = \sigma(s'_2)$ then $s_1 + s'_2 \in G$. We denote this by $H + K \subset G$.

Proposition 1.1: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the open sets under the local homeomorphism σ form a base for the open sets of X . (30, (-))

Proof: Let $G \subset S$ be an open set with $x \in G$. Since σ is a local homeomorphism there is an open set H with $x \in H$ such that

$$\sigma|_H : H \longrightarrow \sigma(H)$$

is a homeomorphism. Then $H \cap G$ is open. So if $x \in H \cap G \subset G$ and

$$\sigma|_{H \cap G} : H \cap G \longrightarrow \sigma(H \cap G)$$

is a homeomorphism, then $\sigma(H \cap G)$ is open in $\sigma(H)$. Hence $\sigma(H \cap G)$ is open in X .

Proposition 1.2: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups then σ is a continuous open map. (95, (-))

Proof: Since σ is a local homeomorphism, it follows that σ is a continuous map. σ is an open map by Proposition 1.1.

Proposition 1.3: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the sheaf space is the union of the stalks of the sheaf. (91, (+))

Proof: It follows that

$$S = \bigcup_{x \in X} S_x$$

from (S_1) of Definition 1.1.

Definition 1.2: A sheaf $\mathcal{A} = (A, \alpha, X)$ of rings with unity over a topological space X is a set A provided with an onto map

$$\alpha : A \longrightarrow X$$

and provided with two structures:

1. Algebraic structure

S_1 : For each $x \in X$ the set $\alpha^{-1}(x) = A_x$ is a ring with unity, denoted 1_x .

2. Topological structure

The sheaf space A is a topological space such that:

S_2 : α is a local homeomorphism of A onto X .

S_3 : The composition laws (addition, multiplication, unity) of the stalks are continuous on A .

Consider the function $f(x) = \frac{1}{x}$ on the interval $[1, 2]$. The function is continuous and differentiable on this interval. The derivative is $f'(x) = -\frac{1}{x^2}$. The function is concave up on this interval.

Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. The function $f(x)$ is concave up on the interval $[1, 2]$. The function $g(x)$ is concave down on the interval $[1, 2]$.

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Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$.

Remark 1.2: Continuity of multiplication means that

$$g : A + A \longrightarrow A$$

defined by

$$g(a, a') = a \cdot a'$$

is continuous. Continuity of unity means that

$$h : X \longrightarrow A$$

defined by

$$h(x) = 1_x$$

is continuous.

Definition 1.3: A sheaf $\mathcal{S} = (S, \sigma, X)$ of (unitary left) $\mathcal{A}$ -modules over a topological space X , where $\mathcal{A} = (A, \alpha, X)$ is a sheaf of rings with unity, is a set S provided with an onto map

$$\sigma : S \longrightarrow X$$

and provided with two structures:

1. Algebraic structure

S_1 : For each $x \in X$ the set $\sigma^{-1}(x) = S_x$ is a (unitary left) A_x -module.

2. Topological structure

The sheaf space S is a topological space such that:

S_2 : σ is a local homeomorphism of S onto X .

S_3 : The composition laws (addition, multiplication by elements of A_x) of the stalks are continuous on S .

Remark 1.3: In the definition of a sheaf, the base space is not assumed to satisfy any separation axioms.

Remark 1.4: Continuity of multiplication by elements of A_x means that

$$g : A + S \longrightarrow S$$

defined by $g(a, s) = a \cdot s$ is continuous, where $A + S$ is a subspace of $A \times S$ consisting of (a, s) for which $\alpha(a) = \sigma(s)$.

Theorem 1.1: If $\mathcal{A} = (A, \alpha, X)$ is a sheaf of rings with unity, then $\mathcal{A}$ is a sheaf of $\mathcal{A}$ -modules. (91, (-))

Proof: We define the product $a \cdot a'$ with $a, a' \in A_x$ as the product $a \cdot a'$ in A_x .

Remark 1.5: Theorem 1.1 shows that sheaves of rings with unity, and hence sheaves of abelian groups, can be considered as special cases of sheaves of $\mathcal{A}$ -modules.

Proposition 1.4: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules, then the stalks have the discrete topology. (95, (-))

Proof: Since σ is a local homeomorphism, then the induced topology on the stalk is the discrete topology.

Definition 1.4: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of abelian groups. The zero element 0_x of the stalk S_x is said to be continuous if

$$f : X \longrightarrow S$$

is continuous, where $f(x) = o_x$, $x \in X$. (9)

Theorem 1.2: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the zero element of the stalk is continuous. (6 , (0))

Proof: Let $x \in X$ and G be an open set such that $f(x) = o_x \in G$ where

$$f: X \longrightarrow S.$$

Then there is an open set G_1 such that $o_x \in G_1 \subset G$ and

$\sigma|_{G_1}$ is a homeomorphism of G_1 onto an open set $\sigma(G_1)$.

Since $o_x + o_x = o_x$ and addition is continuous, there exist open

sets H, K with $o_x \in H, o_x \in K$ such that $H + K \subset G_1$.

Let $L = G_1 \cap H \cap K$. Then L is open, $o_x \in L$ and $\sigma|_L$ is a

homeomorphism of L onto open $\sigma(L)$. Then $x = \sigma(o_x) \in \sigma(L)$

and if $y \in \sigma(L)$, there exists $z \in L$ with $\sigma(z) = y$. Then

$z \in H, z \in K$ and hence $z + z \in G_1$. But $z \in G_1$ and $\sigma|_{G_1}$ is

1 - 1; hence $z \in S_y \cap G_1$. Since $z + z \in S_y \cap G$, $\sigma(z + z) = y$,

$\sigma(z) = y$, then $z = o_y$. If $y \in \sigma(L)$ then $(\sigma|_L)^{-1}(y) = o_y$

so that $f(\sigma(L)) = L$. Thus x is in an open set $\sigma(L)$, with

$f(\sigma(L)) = L \subset G$. Hence f is continuous.

Proposition 1.5: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the set of the zero elements is an open set. (9 , (-))

Proof: In Theorem 1.2 we proved that each o_x is contained in an open set which consists of zero elements only and which projects homeomorphically onto an open set of X .

Definition 1.5: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of abelian groups. The inverse element $-s$ of the stalk $S_{\sigma(s)}$ is said to be continuous if

$$f : S \longrightarrow S$$

is continuous, where $f(s) = -s$, $s \in S$. (6)

Theorem 1.3: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups then the inverse element of the stalk is continuous. (6, (0))

Proof: Let $s \in S$ and G be an open set such that $f(s) = -s \in G$ where

$$f : S \longrightarrow S.$$

Then there exists an open set L such that $o_{\sigma(s)} \in S$ and consists of zero elements only. Since $s + (-s) = o_{\sigma(s)}$ and addition is continuous, then there exist open sets H, K with $s \in H$, $-s \in K$ and $H + K \subset L$. Hence if $u \in H$, $v \in K$ and $\sigma(u) = \sigma(v)$, then $u + v = o_{\sigma(u)}$, that is, $u = -v$. We may assume that $\sigma|_H$ is a homeomorphism. Let

$$H_1 = (\sigma|_H)^{-1} (\sigma(H) \cap \sigma(K \cap G)).$$

Then $s \in H_1$ and since σ is an open, continuous mapping, then

H_1 is open. So if $u \in H_1$, there exists $v \in K \subset G$ with $\sigma(u) = \sigma(v)$.

Then $v = -u = f(u)$. Thus H_1 is an open set and $f(H_1) \subset G$.

Hence f is continuous.

Corollary 1.3: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then subtraction is continuous; i.e., $f: S + S \longrightarrow S$, where $f(s, s') = s - s'$, is continuous.

Definition 1.6: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of abelian groups, let $Y \subset X$ and let $S|Y = \bigcup_{x \in Y} S_x$ be the sheaf space with the induced topology. The sheaf $\mathcal{S}|Y = (S|Y, \sigma|Y, Y)$ is said to be restricted sheaf of $\mathcal{S}$ over Y . (11)

Proposition 1.6: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups and if $Y \subset X$, then restricted sheaf is a sheaf. (11, (-))

Proof: Since σ is a local homeomorphism of $S \longrightarrow X$, then σ is a local homeomorphism of $S|Y \longrightarrow Y$. For each $x \in Y$, $S_x|Y$ is an abelian group, and the operations are continuous on the subspace.

Definition 1.7: A protosheaf $\mathcal{S} = (S, \sigma, X)$ of $\mathcal{A}$ -modules over a topological space X is a set S provided with an onto map

$$\sigma: S \longrightarrow X$$

and provided with one structure:

1. Algebraic structure

S_1 : For each $x \in X$ the set $\sigma^{-1}(x) = S_x$ is a (unitary left) A_x -module. (57)

Proposition 1.7: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules, then there exists a (unique) protosheaf $\underline{\mathcal{S}}$ of $\mathcal{S}$. (95, (+))

Proof: Given any sheaf of $\mathcal{A}$ -modules, we can form a (unique) protosheaf by disregarding the topology of the sheaf space.

2. Sections of Sheaves

The study of sections is one of the central topics in the theory of sheaves, of which the problem of extension is the most important.

Definition 1.8: A section of a sheaf $\mathcal{S} = (S, \sigma, X)$ over an open set Y of the base space X is a continuous map

$$f : Y \longrightarrow S$$

such that $\sigma \circ f = 1|_Y$, where $1|_Y$ denotes the identity map on the set Y . (11)

Theorem 1.4: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then $x \longrightarrow o_x$ is a section of the sheaf. (6, (+))

Proof: We show that $x \longrightarrow o_x$ is a section. Let $x \in X$ be fixed. Let $u \in S_x$, U, V be any neighborhoods of o_x and let

u be mapped homeomorphically by σ onto a neighborhood W of x (such exists by definition). Suppose for $w \in W$, $u_w = \sigma^{-1}(w) \cap U$ and $v_w = \sigma^{-1}(w) \cap V$. Then $w \longrightarrow u_w + v_w$ is a section over W , equal to v at x . Thus it is equal to $w \longrightarrow u_w + v_w$ in a suitable neighborhood of x , which implies $u_w = o_w$ for w in the neighborhood of x .

Remark 1.6: The section $x \longrightarrow o_x$ defined in Theorem 1.4 is called the zero section. We denote the set of all sections over an open set U of the base space X by $\Gamma(U, \mathcal{S})$.

Proposition 1.8: If f is a section of the sheaf $\mathcal{S} = (S, \sigma, X)$ of abelian groups, then f is an open map. (11, (-))

Proof: This is proved by using the fact that f is continuous and σ is a local homeomorphism.

The following theorem characterizes the sections of the sheaf.

Theorem 1.5: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of abelian groups. A set $G \subset S$ is a section over an open set $U \subset X$ iff G is open and $\sigma|_G$ is a homeomorphism. (99, (-))

Proof: To prove the sufficiency, let $G \subset S$ be open and $\sigma(G) = U$. Since $\sigma|_G$ is a homeomorphism and G is open, it follows U is open. Now define $f : U \longrightarrow G$ by $\sigma(G)(f(u)) = u$ for each $u \in U$ which clearly is continuous. To prove the necessity, let f

be a section over U . Then since f is an open map, $f(U)$ is open. Since

$$f : U \longrightarrow f(U)$$

is 1 - 1, open and continuous, then

$$\sigma|_{f(U)} : f(U) \longrightarrow U$$

is a homeomorphism.

Corollary 1.5: If f is a section over $U \subset X$, then f is a homeomorphism of U onto $f(U)$. (95, (-))

Theorem 1.6: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the sections of the sheaf form a base for the open sets of the sheaf space. (99, (-))

Proof: By Corollary 1.5, the section f over $U \subset X$ is a homeomorphism of U onto $f(U)$ in S . By Proposition 1.1, the open sets of S which project homeomorphically onto open sets of X form a base for the open sets of X .

Proposition 1.9: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the intersection of two sections of the sheaf is a section.

Proof: Let $U \longrightarrow f(U)$ and $V \longrightarrow g(V)$ be sections over $U, V \subset X$. Then

$$f(U) \cap g(V)$$

is open and projects homeomorphically onto an open set of X ;

in fact $\sigma(f(U) \cap g(V)) = \sigma(f(U) \cap \sigma(g(V))) = U \cap V$ since σ is 1 - 1 on $f(U)$ and on $g(V)$ and so $\sigma = (\sigma^{-1})^{-1}$ on these sets.

Theorem 1.7: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then the set of all sections of $\mathcal{S}$ over a non-empty open set U of X forms an abelian group. (11, (-))

Proof: Define $f + g$ by

$$(f + g)(x) = f(x) + g(x) ;$$

so that $\Gamma(U, \mathcal{S})$, the collection of all sections over U , is a set with structure. Since addition, zero and inverse are continuous, the set $\Gamma(U, \mathcal{S})$ is an abelian group.

Proposition 1.10: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of abelian groups. If $f \in \Gamma(U, \mathcal{S})$, then the set

$$\{x : f(x) = o_x\}$$

is open in $U \subset X$. (30, (-))

Proof: Since

$$\{x : f(x) = o_x\} = \sigma(f(U) \cap o(U)) ,$$

where o is the zero section over U , then it follows that the set

$$\{x : f(x) = o_x\}$$

is open in U .

Definition 1.9: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of abelian groups.

The set

$$\text{supp } f = \{x : f(x) \neq o_x\}$$

is said to be the support of the section f . (11)

Proposition 1.11: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups, then supports of sections of $\Gamma(U, \mathcal{S})$ are closed in U . (11, (-))

Proof: Let $f \in \Gamma(U, \mathcal{S})$. Since

$$\text{supp } f = U - \{x : f(x) = o_x\}$$

and

$$\{x : f(x) = o_x\}$$

is open in U , then $\text{supp } f$ is closed in U .

Corollary 1.11: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups and if f is a section over X , then $\text{supp } f$ is a closed subset of X .

An important property of sections of sheaves is stated in the following theorem.

Theorem 1.8: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups and if $f, g \in \Gamma(U, \mathcal{S})$ such that

$$f(x) = g(x)$$

for some $x \in U \subset X$, then

$$f = g$$

on some open neighborhood of x . (95, (0))

Proof: This is a re-statement that

$$\text{supp } f - \text{supp } g$$

is closed. Or let W be a neighborhood of $f(x)$ which is mapped homeomorphically by σ onto a neighborhood of x . There is a neighborhood V of x such that $f(V), g(V) \subset W$. If $y \in V$, then

$$f(y) = g(y)$$

since $\sigma(f(y)) = y = \sigma(g(y))$.

Corollary 1.8: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of abelian groups and if $f, g \in \Gamma(U, \mathcal{S})$, then the $\{x \in X : f(x) = g(x)\}$ is an open set.

Theorem 1.9: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules over a compact topological space X and if $Y \subset X$ is a closed subspace, then a section of the induced sheaf $\mathcal{S}|_Y$ extends to a section of the sheaf $\mathcal{S}$. ($?, (-)$)

Proof: Let f be a section of the sheaf $\mathcal{S}|_Y$. By Tietze extension theorem, for each $x \in X$ there exists an open set G such that $x \in G$ (since a section of a sheaf is locally the graph of a continuous map), and there exists

$$t \in \Gamma(G, \mathcal{S}),$$

set of all sections of $\mathcal{S}$ restricted to G , such that t and f coincide on $G \cap Y$. Since X is compact, there exists a finite open covering

$$\{G_i\}$$

where $i = 1, \dots, n$ with $t_i \in \Gamma(G_i, \mathcal{S})$. Now t_i and f coincide on $G_i \cap Y$. Next consider a set of functions $\{g_i\}$ such that

$$\text{supp } g_i \subset G_i.$$

Then we define a section f_i of $\mathcal{S}$ by defining

$$f_i(x) = \begin{cases} g_i(x) \cdot t_i(x) & \text{for } x \in G_i \\ 0 & \text{for } x \notin G_i \end{cases}$$

So that $\sum_{i=1}^n f_i$ is a section of $\mathcal{S}$. Hence a section of $\mathcal{S}|_Y$ can be extended to a section of $\mathcal{S}$.

Remark 1.7: A sheaf is often introduced in practice as the union of its stalks. Since the sections form a base for the open sets of the sheaf space, the topology of the sheaf space is then defined by specifying the sections. In the applications of sheaves, the stalks are not chosen arbitrarily but are defined in some natural way in terms of properties of the base space.

A sheaf may be described by specifying its sections in the following way. Suppose that we are given a space X and mutually disjoint abelian groups S_x (This can always be done by replacing S_x by an isomorphic copy of S_x), one for each $x \in X$. Suppose that we are given a collection $F = \{f\}$ of maps with domain open in X and values in $\bigcup_{x \in X} S_x$; that is,

$$f : \text{dom}(f) \longrightarrow \bigcup_{x \in X} S_x$$

such that if $x \in \text{dom}(f)$ then $f(x) \in S_x$. Suppose that

- (1) the images for all $f \in F$ cover $\bigcup_{x \in X} S_x$.
- (2) if $f_1(x)$, $f_2(x)$ are defined then, for some open set U with $x \in U \subset \text{dom}(f_1) \cap \text{dom}(f_2)$, $(f_1 + f_2)|_U \in F$.
- (3) if $f(x) = o_x$ then, for some open set U with $x \in U \subset \text{dom}(f)$, $f(U)$ consists of zeros only.

Theorem 1.10: If $S = \bigcup_{x \in X} S_x$ with $\{f(U)\}$, for all $f \in F$ and open $U \subset \text{dom}(f)$, as base for the open sets and if

$$\sigma(s) = x$$

for $s \in S_x$, then $\mathcal{S} = (S, \sigma, X)$ is a sheaf. (11, (0))

Proof: Let $s \in f(U) \cap f_1(U_1)$ and let $x = \sigma(s)$. By (1) there exists f_2 with $f_2(x) = -s$. By (2) there exists a neighborhood V of x with $(f + f_2)|_V \in F$. By (3), since $(f + f_2)(x) = s - s = o_x$, there is a smaller neighborhood V' of x with $(f + f_2)(V')$ consisting of zeros. Similarly there is a neighborhood V'_1 of x with $(f_1 + f_2)(V'_1)$ consisting of zeros. Let $W = V' \cap V'_1$. Then

$$(f + f_1 + f_2)|_W = f|_W = f_1|_W.$$

Hence $f(W)$ is in the base, and $s \in f(W) \subset f(U) \cap f_1(U_1)$.

Therefore the axioms for a base are satisfied.

Then $f : U \longrightarrow S$ is continuous. For if $x \in \text{dom}(f)$

and $s = f(x)$, a neighborhood G of s contains a neighborhood $f_1(U_1)$. Again there is an open set W with $s \in f(W) = f_1(W) \subset f(U_1) \subset G$. Hence $f : U \longrightarrow f(U)$ is a homeomorphism, since f is 1-1 and open. Then $\sigma|f(U)$ is the inverse homeomorphism, and so its image U is open. Thus σ is a local homeomorphism.

Addition is continuous. For if $s, s' \in S_x$ with $s + s' \in f_2(U_2)$ suppose $s \in f(U)$, $s' \in f_1(U_1)$. By (2) there is a neighborhood V of x with $(f_1 + f_2)|V \in F$. Then $s + s' \in f_2(U_2) \cap (f + f_1)(V)$, and hence for some W , with $x \in W \subset U_2 \cap V$, $f_2|W = (f + f_1)|W$. Thus if $u \in f(W)$, $v \in f_1(W)$ and $\sigma(u) = \sigma(v)$, therefore $f(u) + f_1(v) = f_2(v) \in f_2(U_2)$.

3. Homomorphisms, Subsheaves and Quotient Sheaves

Definition 1.10: Let $\mathcal{A} = (A, \alpha, X)$ be a sheaf of rings with unity and $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ be sheaves of $\mathcal{A}$ -modules over the same base space X . A sheaf homomorphism

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

is a continuous map

$$h : S \longrightarrow R$$

such that

$$H_1 : \rho \cdot h = \sigma.$$

$$H_2 : \text{Restriction } h|_{S_x} = h_x : S_x \longrightarrow R_x \text{ is an}$$

$$A_x\text{-homomorphism for each } x \in X. \quad (11)$$

Remark 1.8: This definition includes as a special case the definition of homomorphisms of sheaves of abelian groups and sheaves of A -modules.

Proposition 1.12: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are sheaves of A -modules and if h is a sheaf homomorphism

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

then the image of each section is a section. (30, (-))

Proof: If $f : U \longrightarrow S$ is a section, then

$$hf : U \longrightarrow R ,$$

the image of the section f defined by $(hf)(x) = h(f(x))$, is continuous and $\rho(hf)(x) = \rho h(f(x)) = \sigma f(x)$ since h is a sheaf homomorphism so that $\rho(hf)(x) = x$ since $f \in \Gamma(U, \mathcal{S})$.

Therefore $\rho(hf) = 1|U$.

Proposition 1.13: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are sheaves of A -modules, then a sheaf homomorphism

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

is an open map and a local homeomorphism of the topological spaces S and R . (99, (-))

Proof: h is an open map. For if $G \subset S$ is an open set then $h(G)$ is a union of sections of $\mathcal{R}$. Hence h is an open map. h is a local homeomorphism. For each element of S is contained in some section $f(U)$, $U \subset X$. So $hf(U)$ is a section in $\mathcal{R}$ and

$$h|f(U) = hf\sigma|f(U) .$$

Hence each of the maps

$$\sigma|_{f(U)} : f(U) \longrightarrow U$$

and

$$hf : U \longrightarrow hf(U)$$

is a homeomorphism.

Theorem 1.11: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are sheaves of $\mathcal{A}$ -modules and if f, g are sheaf homomorphisms of $\mathcal{S}$ to $\mathcal{R}$, then the maps, $u \in S, a \in \mathcal{A}_x$,

$$f + g : S \longrightarrow R \quad \text{and} \quad af : S \longrightarrow R,$$

defined by $(f + g)_u = f_u + g_u$ and $(af)_u = af_u$ respectively, are sheaf homomorphisms and the set of all sheaf homomorphisms under this addition and ring multiplication is a module. (91, (-))

Proof: Let $h = f + g$. Then h satisfies the commutativity $\sigma = \rho \cdot h$. Since addition of two continuous maps is continuous, then

$$\sigma = \rho \cdot (f + g) = \rho \cdot h.$$

Now h is a module homomorphism. For if $u, v \in S_x$ then

$$\begin{aligned} (f + g)_{(u+v)} &= f_{(u+v)} + g_{(u+v)} = f_u + f_v + g_u + g_v \\ &= f_u + g_u + f_v + g_v = (f + g)_u + (f + g)_v \end{aligned}$$

and if $a \in \mathcal{A}_x$

$$((f + g)a)_u = (fa + ga)_u = af_u + ag_u = a(f_u + g_u) = a(f + g)_u.$$

We show that $\text{Hom}(\mathcal{S}, \mathcal{R})$ has a $\mathcal{A}_x$ -module structure. Now it

is clear that $f + g = g + f$ for $g, f \in \text{Hom}(\mathcal{S}, \mathcal{R})$. If

$f, g, k \in \text{Hom}(\mathcal{S}, \mathcal{R})$, then $((f + g) + k)_u = f_u + g_u + k_u = (f + (g + k))_u$.

There is a zero element. For if f_0 is the constant map of value 0 of $\mathcal{S}$ and $\mathcal{R}$, then $f + f_0 = f_0 + f$ for $f \in \text{Hom}(\mathcal{S}, \mathcal{R})$.

If $f \in \text{Hom}(\mathcal{S}, \mathcal{R})$ then the map

$$f' : u \longrightarrow -f_u$$

is a module homomorphism, as can be verified. Since

$f + f' = f' + f = f_0$, then $\text{Hom}(\mathcal{S}, \mathcal{R})$ is an (additive) abelian group. So that for $a, a', a'' \in A_x$ and $f, g \in \text{Hom}(\mathcal{S}, \mathcal{R})$.

$$(a(f + g))_u = (af + ag)_u = af_u + ag_u$$

$$(a' + a'')f_u = ((a' + a'')f)_u = (a'f + a''f)_u = a'f_u + a''f_u$$

$$a'(a''f_u) = a'((a''f)_u) = (a'a''f)_u = (a'a'')f_u$$

$$1 \cdot f_u = (1 \cdot f)_u = f_u$$

This shows that $\text{Hom}(\mathcal{S}, \mathcal{R})$ has a structure of an A_x -module.

Definition 1.11: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of $\mathcal{A}$ -modules and $S' \subset S$. $\mathcal{S}' = (S', \sigma|_{S'}, X)$ is said to be a subsheaf of the sheaf $\mathcal{S} = (S, \sigma, X)$ if

SS_1 : S' is an open set in S .

SS_2 : The stalk $S'_x = S' \cap S_x$ is a sub- A_x -module of S_x . (11)

Proposition 1.14: If $\mathcal{S}' = (S', \sigma', X)$ and $\mathcal{S} = (S, \sigma, X)$ are two sheaves of $\mathcal{A}$ -modules with $S' \subset S$, then the inclusion map

$$i : S' \longrightarrow S$$

is a sheaf homomorphism.

Proof: Since $i : S' \longrightarrow S$ is a continuous map and

$$\sigma' = \sigma \circ i = \sigma|_{S'}$$

and

$$i|_{S'_x} : S'_x \longrightarrow S_x$$

is the inclusion homomorphism of the sub- A_x -module S'_x , it follows that i is a sheaf homomorphism.

Theorem 1.12: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules, then the set of zeros in $\mathcal{S}$ is a subsheaf. (30, (-))

Proof: The set of zeros $\{o_x\}$ is an open set in S , and o_x is a sub- A_x -module of S_x . Hence by Definition 1.11, the set of all zeros is a subsheaf.

Remark 1.9: The subsheaf consisting of all the zeros of the sheaf is called the zero sheaf, and is denoted by O -sheaf.

Theorem 1.13: If $\mathcal{S}' = (S', \sigma', X)$ and $\mathcal{S} = (S, \sigma, X)$ are two sheaves of $\mathcal{A}$ -modules with $S' \subset S$, then there is at most one structure on S' such that $\mathcal{S}'$ is a subsheaf. (95, (-))

Proof: We show the existence of a subsheaf. For each $s \in S'$ there exists an open set G , $s \in G \subset S'$, with $\sigma|_G$ a homeomorphism. Then $G \cap S'$ is open in S' and

$$(\sigma|_{S'})|_{G \cap S'} = \sigma|_{G \cap S'}$$

is a local homeomorphism. By Proposition 1.14, the inclusion map

$$i : S' \longrightarrow S$$

is a sheaf homomorphism. Then S' is an open subset of S since i is an open map. Further the topology on S' is the one induced from S . Then

$$\sigma' = \sigma \cdot i = \sigma|_{S'}$$

and since

$$i|_{S'_x} : S'_x \longrightarrow S_x$$

is a homomorphism, S'_x is a sub- A_x -module of S_x . Now the operations which are continuous in S are continuous in the subspace S' . Therefore $\mathcal{S}' = (S', \sigma'|_{S'}, X)$ is a sheaf.

Theorem 1.14: If $\mathcal{S}' = (S', \sigma', X)$ is a subsheaf of $\mathcal{S} = (S, \sigma, X)$ of $\mathcal{A}$ -modules, then $\mathcal{S}'' = \mathcal{S}/\mathcal{S}'$ is a sheaf of $\mathcal{A}$ -modules. (95, (6))

Proof: Form a protosheaf $\underline{S}/\underline{S}'$ with projection $\sigma'' : (\underline{S}/\underline{S}')_x \longrightarrow S_x/S'_x$ that is define

$$\sigma''(\underline{S}/\underline{S}')_x = x.$$

Define a homomorphism

$$h : S \longrightarrow \underline{S}/\underline{S}'$$

be letting $h|_{S_x}$ be the natural map

$$h|_{S_x} : S_x \longrightarrow S_x/S'_x$$

Let $\underline{S}/\underline{S}'$ be the protosheaf $\underline{S}/\underline{S}'$ together with the identification topology given by h . The following diagram is commutative.

$$\begin{array}{ccc} S & \xrightarrow{h} & \underline{S}/\underline{S}' \\ \sigma \downarrow & \searrow \sigma'' & \\ X & & \end{array}$$

Let $y \in S/S'$, $h(z) = y$ and $\sigma(z) = x$. Let U be a neighborhood of z which is mapped homeomorphically by σ onto a neighborhood V of x . Since h is an open map, then $h(U)$ is a neighborhood of y . But $h|_U$ is 1 - 1 for no two elements of U belong to the same stalk, and so $h|_U$ is a homeomorphism. Therefore

$$\sigma'' : h(U) \longrightarrow V$$

is a homeomorphism; that is, σ'' maps $h(U)$ homeomorphically onto V , a neighborhood of x .

To prove that addition and multiplication by $a \in A_x$ in S/S' are continuous, we need only consider the following commutative diagrams in which h and $h + h$ are sheaf epimorphisms, and so are identification maps. (For any spaces Y, Z a map $Y \longrightarrow Z$ which is a continuous, open onto map is an identification map.)

$$\begin{array}{ccc} S/S' + S/S' & \longrightarrow & S/S' \\ \downarrow h + h & & \downarrow h \\ S + S & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} S/S' & \longrightarrow & S/S' \\ \downarrow h & & \downarrow h \\ S & \longrightarrow & S \end{array}$$

Proposition 1.15: If $\mathcal{S}' = (S', \sigma', X)$ and $\mathcal{R}' = (R', \rho', X)$ are subsheaves of $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$, respectively, such that $h(S') \subset R'$, where $h : \mathcal{S} \longrightarrow \mathcal{R}$, then there are induced homomorphisms

$$\begin{aligned} \text{and} \quad h' : \mathcal{S}' &\longrightarrow \mathcal{R}' \\ h'' : \mathcal{S}/\mathcal{S}' &\longrightarrow \mathcal{R}/\mathcal{R}' \end{aligned}$$

$$\text{with} \quad h i = i h'$$

$$\text{and} \quad h'' j = j h$$

where i denotes the inclusion homomorphism and j denotes the natural homomorphism of a sheaf onto a quotient sheaf.

Proof: The result is clear for stalks, and the fact that h', h'' are homomorphisms of sheaves follows from the fact that they are continuous.

4. Exact Sequence of Sheaves

Definition 1.12: Let $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ be any sheaves. A homomorphism

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

is said to be a monomorphism if

$$\ker h = 0,$$

an epimorphism if

$$\operatorname{im} h = R$$

and an isomorphism if

$$\ker h = 0 \text{ and } \operatorname{im} h = R. \quad (91)$$

Remark 1.10: The quotient sheaves $R/\operatorname{im} h$ and $S/\ker h$ are called the cokernel of h and the coimage of h respectively.

Definition 1.13: A sequence of homomorphisms of sheaves

$$\cdots \longrightarrow \mathcal{S}_{n-1} \xrightarrow{h_n} \mathcal{S}_n \xrightarrow{h_{n+1}} \mathcal{S}_{n+1} \longrightarrow \cdots$$

is said to be exact at $\mathcal{S}_n$ if

$$\ker h_{n+1} = \operatorname{im} h_n.$$

A sequence is said to be an exact sequence if it is exact

at each $\mathcal{S}_n$. (91)

Proposition 1.16: If

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

is a homomorphism of sheaves, then the sequence

$$0 \longrightarrow \ker h \xrightarrow{i} \mathcal{S} \xrightarrow{h'} \operatorname{im} h \longrightarrow 0$$

is exact, where i is the inclusion homomorphism and h' is defined by

$$h'(u) = h(u)$$

for each u in the sheaf space of $\mathcal{S}$.

Proof: The statement is the composite of the following three trivial statements:

(a) $i : \ker h \longrightarrow \mathcal{S}$ is a monomorphism.

(b) $\ker h' = \ker h$.

(c) $h' : \mathcal{S} \longrightarrow \operatorname{im} h$ is an epimorphism.

Theorem 1.15: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are sheaves and h is a sheaf homomorphism

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

then the image of h and the kernel of h are sheaves. (91, (-))

Proof: Let

$$\ker h = \{u \in S : h_x(u) = o_x, u \in S_x\}$$

which is the subset of S mapped into the neutral elements of R .

The $\ker h$ is an open set. For the set of neutral elements of R

is the image of the zero section of R over X , and this is an

open set in R . Since h is continuous, then the preimage of the

set of neutral elements of R is open in S .

Now

$$\ker h \cap S_x$$

is the kernel of the homomorphism $h|_{S_x}$, and thus is a substructure of S_x . Therefore $\ker h$ is a subsheaf of $\mathcal{S}$.

Now we show that the image of h , denoted $\text{im } h$, is a sheaf. That $\text{im } h$ is open follows from the continuity of h , the commutativity of h and the projection map, that is,

$$\rho h = \sigma,$$

and the fact that σ and ρ are local homeomorphisms. As before

$$\text{im } h \cap R_x$$

is a substructure of R_x . So that $\text{im } h$ is a subsheaf of the sheaf $\mathcal{R}$.

Corollary 1.15: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are sheaves and if

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

then the cokernel and the coimage are sheaves.

Proof: The quotient sheaves

$$R/\text{im } h$$

and

$$S/\ker h$$

are the cokernel and coimage of h , respectively.

5. Operations on Sheaves

Definition 1.14: Let $\mathcal{A} = (A, \alpha, X)$ be a sheaf of rings with unity and let

$$\{\mathcal{S}^i\}_{i \in I}$$

be a system of sheaves of $\mathcal{A}$ -modules. Let $\mathcal{S}$ be the unique sheaf whose stalks are the direct sums

$$\bigsqcup_{i \in I} S_x^i$$

such that the homomorphisms

$$h_x^j : S_x^j \longrightarrow S_x$$

determine a sheaf homomorphism

$$h^j : \mathcal{S}^j \longrightarrow \mathcal{S}.$$

The sheaf $\mathcal{S}$ is said to be the direct sum of the sheaves $\mathcal{S}^i$ and is denoted

$$\mathcal{S} = \bigsqcup_{i \in I} \mathcal{S}^i. \quad (91)$$

Theorem 1.16: If

$$\{\mathcal{S}^i\}_{i \in I}$$

is a system of sheaves of $\mathcal{A}$ -modules, then there is a unique sheaf $\mathcal{S}$ whose stalks are the direct sums $\bigsqcup_{i \in I} S_x^i$ such that

the homomorphisms

$$h_x^j : S_x^j \longrightarrow S_x$$

determine a sheaf homomorphism

$$h^j : \mathcal{S}^j \longrightarrow \mathcal{S}. \quad (91, (0))$$

Proof: We prove the uniqueness. If

$$s = \bigsqcup_{i \in I} s^i \in \bigsqcup_{i \in I} S_x^i ,$$

with $s^i = 0$ except for $i_1, \dots, i_k$, choose a neighborhood U of x and sections

$$f_j : U \longrightarrow S^{i_j}$$

where $j = 1, \dots, k$ such that $f_j(x) = s^{i_j}$. Let

$$f : U \longrightarrow S$$

where $S = \bigcup_{x \in X} \bigsqcup_{i \in I} S_x^i$ be defined by

$$f(y) = \bigsqcup_j h^{i_j} f_j(y) .$$

Since h^{i_j} is a sheaf homomorphism, the composite function

$$U \xrightarrow{f_j} S^{i_j} \xrightarrow{h^{i_j}} S$$

is a section and hence f is a section. Since

$$f(x) = \bigsqcup_j h^{i_j} f_j(x) = \bigsqcup_j h^{i_j} s^{i_j} = \bigsqcup_i s^i = s ,$$

the section passes through s . Thus, since such sections cover S , they uniquely determine the topology of S .

Now we prove the existence. Form the presheaves of sections

$$(\bar{A}, \bar{\mathcal{S}}^i) = \{A_U, S_U^i, p_{VU}', p_{VU}^i\}$$

where $A_U = \Gamma(U, A)$ and $S_U^i = \Gamma(U, \mathcal{S}^i)$. Then

$$\{A_U, \bigsqcup_i S_U^i, p_{VU}', p_{VU}\} ,$$

where U runs thru the directed set of all the open sets of X , is a presheaf. This presheaf determines a sheaf $(A, \mathcal{S})$.

So there is a homomorphism

$$h_U^j : s_U^j \longrightarrow \varinjlim_i s_U^i$$

with commutativity in

$$\begin{array}{ccc} (A_U, s_U^j) & \xrightarrow{h_U^j} & (A_U, \varinjlim_i s_U^i) \\ (p_{VU}^j, p_{VU}) \downarrow & & \downarrow (p_{VU}^j, p_{VU}) \\ (A_V, s_V^j) & \xrightarrow{h_V^j} & (A_V, \varinjlim_i s_V^i) \end{array}$$

Hence there is an induced sheaf homomorphism

$$h^j : \mathcal{S}^j \longrightarrow \mathcal{S}.$$

The stalk (A_x, S_x) of $(A, \mathcal{S})$ is the direct limit of the direct system $\{A_U, \varinjlim_i s_U^i, p_{VU}^j, p_{VU}\}_{x \in U}$ which is identified with the direct sum of the direct limits

$$(A_x, S_x^i) = \varinjlim \{A_U, s_U^i, p_{VU}^j, p_{VU}^i\}_{x \in U}.$$

Thus

$$S_x = \varinjlim_i S_x^i$$

Hence the required sheaf exists.

Remark 1.11: There is also a homomorphism $t_U^j : \varinjlim_i s_U^i \longrightarrow s_U^j$,

defined by $t_U^j(\varinjlim_i s^i) = s^j$, with commutativity in

$$\begin{array}{ccc} (A_U, \varinjlim_i s_U^i) & \xrightarrow{t_U^j} & (A_U, s_U^j) \\ (p_{VU}^j, p_{VU}^i) \downarrow & & \downarrow (p_{VU}^j, p_{VU}^j) \\ (A_V, \varinjlim_i s_V^i) & \xrightarrow{t_V^j} & (A_V, s_V^j) \end{array}$$

Hence there is a sheaf homomorphism $t^j : \mathcal{S} \longrightarrow \mathcal{S}^j$ which is onto. If $s = \bigsqcup_i s^i \in S$, then $t^j(s) = s^j$, so $t^j h^j(s^j) = s^j$ and $t^j h^i(s^j) = 0$ if $i \neq j$.

Proposition 1.17: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are sheaves and if h^i are the homomorphism

$$h^i : \mathcal{S}^i \longrightarrow \mathcal{S} = \bigsqcup_{i \in I} \mathcal{S}^i$$

and g_i are the homomorphisms

$$g_i : \mathcal{S}^i \longrightarrow \mathcal{R},$$

then there is an induced homomorphism

$$g : \bigsqcup_{i \in I} \mathcal{S}^i \longrightarrow \mathcal{R}$$

with

$$gh^i = g_i. \quad (91, (-))$$

Proof: If

$$s = \bigsqcup_i s^i \in S_x$$

let

$$g(s) = \bigsqcup_i g_i(s^i) \in R_x.$$

Then

$$g|_{S_x} : S_x \longrightarrow R_x$$

is clearly a homomorphism. Choose an open set U and sections

$$f_j : U \longrightarrow S^{i_j}$$

so that the section defined by

$$f(y) = \bigsqcup_i h^{i_j} f_j(y)$$

goes thru s . Then

$$gf(y) = \bigsqcup_i g_{i_j} f_j(y)$$

and gf being the sum of a finite number of sections is a section. Thus g is continuous and is a sheaf homomorphism.

Definition 1.15: Let $\mathcal{A} = (A, \alpha, X)$ and $\mathcal{B} = (B, \beta, X)$ be sheaves of commutative rings with unity, let $\mathcal{R} = (R, \rho, X)$ and $\mathcal{S} = (S, \sigma, X)$ be sheaves of $\mathcal{A}$ -modules and let $\mathcal{I} = (T, \tau, X)$ be a sheaf of $\mathcal{B}$ -modules. A bihomomorphism

$$f : (\mathcal{A}, \mathcal{R}, \mathcal{S}) \longrightarrow (\mathcal{B}, \mathcal{I})$$

consists of maps

$$f' : A \longrightarrow B$$

and

$$f'' : R + S \longrightarrow T$$

which commutes with projections such that, for each $x \in X$, the restriction

$$f_x : (A_x, R_x, S_x) \longrightarrow (B_x, T_x)$$

is a bihomomorphism. (11)

Definition 1.16: Let $\mathcal{A} = (A, \alpha, X)$ be a sheaf of commutative rings with unity. Let $\mathcal{R} = (R, \rho, X)$ and $\mathcal{S} = (S, \sigma, X)$ be sheaves of $\mathcal{A}$ -modules. The sheaf $\mathcal{Q} = (Q, \varphi, X)$ of $\mathcal{A}$ -modules together with a bihomomorphism

$$q : (\mathcal{R}, \mathcal{S}) \longrightarrow \mathcal{Q},$$

with $\text{im } q_x$ generating Q_x for each x such that, if

$$f : (\mathcal{A}, \mathcal{R}, \mathcal{S}) \longrightarrow (\mathcal{B}, \mathcal{I})$$

is a bihomomorphism, there is a unique homomorphism

$$\bar{f} : (\mathcal{A}, \mathcal{Q}) \longrightarrow (\mathcal{B}, \mathcal{I})$$

with $\bar{f}_q = f$, is said to be the tensor product

$$R \otimes_A S$$

of the sheaves R and S over A and is unique upto isomorphism. (11)

Remark 1.12: Similarly, each Q_x together with a homomorphism

$$q_x : (R_x, S_x) \longrightarrow Q_x$$

is the tensor product

$$R_x \otimes_{A_x} S_x$$

of the stalks R_x and S_x .

Theorem 1.17: If $A = (A, X)$ is a sheaf of commutative rings with unity, and $R = (R, \rho, X)$ and $S = (S, \sigma, X)$ are sheaves of A -modules, then there exists a sheaf Q of A -modules and a homomorphism

$$q : (R, S) \longrightarrow Q$$

with $\text{im } q_x$ generating Q_x for each x such that, if

$$f : (A, R, S) \longrightarrow (B, T)$$

is a bihomomorphism, where $B = (B, \beta, X)$ is a sheaf of commutative rings with unity and $T = (T, \tau, X)$ is a sheaf of B -modules, then there is a unique homomorphism

$$\bar{f} : (A, Q) \longrightarrow (B, T)$$

with

$$\bar{f}_q = f. \quad (11, (-))$$

Proof: The proof is not given.

1. Definition of a Presheaf

A sheaf may be considered as a special case of the more general notion of a presheaf.

Definition 2.17: A presheaf $\mathcal{P} = \{A_U, S_U, p_{VU}\}$ over a topological space X is a collection of $\mathcal{A}$ -modules

$$\{A_U, S_U\}$$

indexed by the open sets $U \subset X$ such that if $V \subset U$ there exists a homomorphism

$$p_{VU} : (A_U, S_U) \longrightarrow (A_V, S_V)$$

with the following properties:

$$P_1 : (A_\emptyset, S_\emptyset) = (0, 0), \text{ where } \emptyset \text{ is the empty set.}$$

$$P_2 : p_{UU} \text{ is the identity.}$$

$$P_3 : \text{ If } W \subset V \subset U \text{ then } p_{WU} = p_{WV}p_{VU}. \quad (54)$$

Remark 2.13: For any topological space the open sets form a directed set, so the definition of a presheaf may be stated as follows: A presheaf of $\mathcal{A}$ -modules over a topological space X is a direct system

$$\{A_U, S_U, p_{VU}\}$$

indexed by the directed set of open sets of X such that

$$(A_\emptyset, S_\emptyset) = (0, 0).$$

2. Presheaf of Sections

Theorem 2.18: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules, then the sections of the sheaf form a presheaf. (30, (-))

Proof: For each open set $U \subset X$ the set $\Gamma(U, \mathcal{S})$ is a (unitary left) $\Gamma(U, \mathcal{A})$ -module. If $V \subset U$, let

$$p_{VU} : (\Gamma(U, \mathcal{A}), \Gamma(U, \mathcal{S})) \longrightarrow (\Gamma(V, \mathcal{A}), \Gamma(V, \mathcal{S}))$$

denote the restriction homomorphism. Since $\Gamma(\emptyset, \mathcal{S}) = 0$ and $(\emptyset, \mathcal{A}) = 0$, then the system

$$\{ \Gamma(U, \mathcal{A}), \Gamma(U, \mathcal{S}), p_{VU} \}$$

is a presheaf.

Remark 2.14: The presheaf in Theorem 2.18 is called the presheaf of sections of a sheaf $(\mathcal{A}, \mathcal{S})$ and is denoted by $(\bar{\mathcal{A}}, \bar{\mathcal{S}})$.

Remark 2.15: If $\mathcal{P} = \{S_U, p_{VU}\}$ is a presheaf of abelian groups over a topological space X , elements in S_U are called sections of $\mathcal{P}$ on U . Elements in S_X are called global sections.

For $x \in X$ we set

$$S_x = \varinjlim_{x \in U} S_U = \text{stalk of } \mathcal{P} \text{ at } x$$

where $\varinjlim_{x \in U} S_U$ is the direct limit taken over the set of all neighborhoods U of x in X .

Definition 2.18: Let $\mathcal{P} = \{A_U, S_U, p_{VU}\}$ be a presheaf of $\mathcal{A}$ -modules and let (A_x, S_x) denote the direct limit of the system. Let

$$p_{xU} : (A_U, S_U) \longrightarrow (A_x, S_x)$$

be the homomorphism which sends each element into its equivalence class. If $a \in A_U$, $s \in S_U$ the image

$$p_{xU}a = a_x$$

$$p_{xU}s = s_x$$

is said to be the germ of the section a at x and of the section s at x , respectively. (2)

Theorem 2.18: If $\mathcal{P} = \{A_U, S_U, p_{VU}\}$ is a presheaf of $\mathcal{A}$ -modules, then the sections $\{a_U\}$ and $\{s_U\}$ form a base for the open sets of A and S , respectively. (30, (0))

Proof: Since $\{a_U\}$ covers A and if $u \in a_U \cap b_U$, $x = \alpha(u)$, where $\alpha : A \longrightarrow X$, then

$$u = a_x = b_x$$

with $a \in A_U$, $b \in A_V$ and $x \in U \cap V$. Then for some W with $x \in W \subset U \cap V$, $p_{WU}a = p_{WV}b = c$, say. Since

$$p_{xU}a = p_{xW}p_{WU}a = p_{xW}c = p_{xW}p_{WV}b = p_{xV}b$$

for each $x \in W$, then $c_W = a_W = b_W$. So

$$u \in c_W = a_W = b_W = a_U \cap b_V$$

which shows that $\{a_U\}$ is a base for the open sets in A .

Similarly the sections $\{s_U\}$ form a base for the open sets in S .

3. Homomorphism of Presheaves

Definition 2.19: Let $\mathcal{P} = \{A_U, S_U, p_{VU}\}$ and $\mathcal{P}' = \{A'_U, S'_U, p'_{VU}\}$ be presheaves over the topological space X . A homomorphism of presheaves

$$f : \mathcal{P}' \longrightarrow \mathcal{P}$$

is a collection $\{f_U\}_{U \subset X}$ of homomorphisms

$$f_U : (A'_U, S'_U) \longrightarrow (A_U, S_U)$$

such that $f_V p'_{VU} = p_{VU} f_U$. (30)

4. Exact Sequence of Presheaves

Definition 2.20: A sequence $\mathcal{P}' \xrightarrow{f} \mathcal{P} \xrightarrow{g} \mathcal{P}''$ of homomorphisms of presheaves of $\mathcal{A}$ -modules is said to be exact if

$$\text{im } f = \ker g ;$$

i.e., if the sequence $(A'_U, S'_U) \xrightarrow{f_U} (A_U, S_U) \xrightarrow{g_U} (A''_U, S''_U)$ is exact for each U .

Theorem 2.19: If $\mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}''$

is an exact sequence of homomorphisms of presheaves, then the induced sequence

$$\mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}''$$

of sheaves is exact. (30, (-))

Proof: The sequences $(A_x, S'_x) \longrightarrow (A_x, S_x) \longrightarrow (A_x, S''_x)$ are exact by a property of the direct limit.

5. Sheaf Induced by a Presheaf

Remark 2.16: We will denote by

$$\bar{a} : U \longrightarrow \bigcup_{x \in X} A_x$$

$$\bar{s} : U \longrightarrow \bigcup_{x \in X} S_x$$

the function for which

$$\bar{a}(x) = a_x$$

$$\bar{s}(x) = s_x$$

respectively. For each $V \subset U$ we write

$$a_V = \bar{a}(V) = \{a_x : x \in V\}$$

$$s_V = \bar{s}(V) = \{s_x : x \in V\}$$

respectively.

Proposition 2.18: Let $A = \bigcup_{x \in X} A_x$, $S = \bigcup_{x \in X} S_x$. Define

$$\alpha : A \longrightarrow X, \quad \sigma : S \longrightarrow X$$

by $\alpha(a_x) = x$ and $\sigma(s_x) = x$, respectively. Then

$$\alpha \bar{a} : U \longrightarrow U, \quad \sigma \bar{s} : U \longrightarrow U$$

are the identity maps on U .

Proof: This is clear from Remark 2.16.

Theorem 2.20: If $\mathcal{P} = \{A_U, S_U, p_{VU}\}$ is a presheaf of $\mathcal{A}$ -modules over a topological space X , where $\mathcal{A} = (A, \alpha, X)$ is a sheaf of rings with unity then $\mathcal{S} = (S, \sigma, X)$ is a sheaf $\mathcal{A}$ -modules. (30, (0))

Proof: If $a \in A_U$, then

$$\bar{a} : U \longrightarrow A$$

is continuous. For, if $a_x \in b_V$ with $b \in A_V$, then there exists $c \in A_W$ with $x \in W \subset U \cap V$ such that

$$\bar{a}(W) = a_W = c_W = b_W \subset b_V.$$

Also

$$\bar{a} : U \longrightarrow A$$

is an open map since for open set $V \subset U$,

$$a_V = (p_{VU}a)_V$$

is open by definition. Hence

$$\bar{a} : U \longrightarrow a_U$$

being 1 - 1 is a homeomorphism of a_U and the inverse

$$\alpha|_{a_U} : a_U \longrightarrow U$$

is a homeomorphism of a_U onto the open set U . Similarly,

$$\bar{s} : U \longrightarrow S$$

is continuous and

$$\sigma|_{s_U} : s_U \longrightarrow U$$

is a homeomorphism. Thus α and σ are local homeomorphisms.

For each $x \in X$, the set

$$\alpha^{-1}(x) = A_x$$

is a ring with unity, and

$$\sigma^{-1}(s) = s_x$$

is a (unitary left) A_x -module.

The addition is continuous. For if $a_x, b_x \in A_x$ with $a \in A_U, b \in A_{U_1}, x \in U \cap U_1$ and $a_x + b_x \in c_{U_2}$ with $c \in A_{U_2}$, then, for some W with $x \in W \subset U \cap U_1 \cap U_2$,

$$p_{WU}^a + p_{WU_1}^b = p_{WU_2}^c.$$

Thus $a_x \in a_w, b_x \in b_w$ and for any $a' \in a_w, b' \in b_w$ with $\alpha(a') = \alpha(b') = y$ say, we have

$$a' + b' = a_y + b_y = c_y \in c_{U_2}.$$

The unit is continuous. For if $1_x \in A_x$ is the unit element of A_x and $1_x \in b_U$. Then for some V with $x \in V \subset U$, p_{VU}^b is the unit element of A_V . Then $b_V \subset b_U$ and consists of the unit element 1_y for $y \in V$.

Similarly, the other operations of $\mathcal{A}$ and $\mathcal{S}$ are continuous.

Remark 2.17: The sheaf defined in Theorem 2.20 is called the sheaf induced by the presheaf.

Theorem 2.21: If $\mathcal{S}' = (S', \sigma', X)$ and $\mathcal{S} = (S, \sigma, X)$ are sheaves of $\mathcal{A}$ -modules determined by the presheaves $\mathcal{P}' = \{A'_U, S'_U, p'_{VU}\}$

and $\mathcal{P} = \{A_U, S_U, p_{VU}\}$, then the presheaf homomorphism

$$f : \mathcal{P}' \longrightarrow \mathcal{P}$$

induces a sheaf homomorphism

$$f : \mathcal{S}' \longrightarrow \mathcal{S}. \quad (30, (0))$$

Proof: For each x , $\{f_U\}_{x \in X}$ induces a homomorphism

$$f_x : (A'_x, S'_x) \longrightarrow (A_x, S_x)$$

with $f_x p'_{xU} = p_{xU} f_U$, and these homomorphisms f_x define a map

$$f : (A', S') \longrightarrow (A, S).$$

If for $a' \in A'_U$, $f_U a' = a$, then

$$a_x = p_{xU} a = p_{xU} f_U a' = f_x p'_{xU} a' = f_x(a').$$

Thus $f(a'_U) = a_U$ and f is a local homeomorphism, and hence is continuous. Therefore f is a sheaf homomorphism.

Theorem 2.22: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules and

if $\bar{\mathcal{S}} = \{A_U, S_U, p_{VU}\}$ is its presheaf of sections and if

$\mathcal{S}' = (S', \sigma', X)$ is the sheaf determined by the presheaf $\bar{\mathcal{S}}$, then $\mathcal{S}'$ and $\mathcal{S}$ are canonically isomorphic. (54, (0))

Proof: Denote the sheaf $\mathcal{S}$ by $(\mathcal{A}, \mathcal{S})$, the presheaf of sections $\bar{\mathcal{S}}$ by $(\bar{\mathcal{A}}, \bar{\mathcal{S}})$ and $\mathcal{S}'$ by $(\mathcal{A}', \mathcal{S}')$. We show that the following diagram commutes:

$$\begin{array}{ccc} & (\bar{\mathcal{A}}, \bar{\mathcal{S}}) & \\ \swarrow \text{dashed} & & \searrow \text{dashed} \\ (\mathcal{A}, \mathcal{S}) & \longleftarrow & (\mathcal{A}', \mathcal{S}') \end{array}$$

If $x \in U$ and $f \in \Gamma(U, \mathcal{A})$, let $h_{xU}f = f(x) \in A_x$. Similarly, if $s \in \Gamma(U, \mathcal{S})$, let $h_{xU}s = s(x) \in S_x$.

Then

$$h_{xU} : (\Gamma(U, \mathcal{A}), \Gamma(U, \mathcal{S})) \longrightarrow (A_x, S_x)$$

is a homomorphism and, if $x \in V \subset U$, $p_{VU} = h_{xU}$. Then there is an induced homomorphism

$$h_x : (A'_x, S'_x) \longrightarrow (A_x, S_x)$$

with $h_x p_{xU} = h_{xU}$. Now h_x is an isomorphism. For if $u \in A_x$, then there is some section $f : U \longrightarrow A$ with $f(x) = u$. Then

$$u = f(x) = h_{xU}f = h_x p_{xU}f \in \text{im } h_x,$$

and if $u' \in A'_x$ with $h_x u' = 0$, choose a representative

$f \in \Gamma(U, \mathcal{A})$ for u' . Then $f(x) = h_{xU}f = h_x u' = 0$. Hence, for some V , with $x \in V \subset U$, $f|_V = 0$. Therefore

$$u' = p_{xU}f = p_{xV}p_{VU}f = p_{xV}0_V = 0.$$

Thus

$$h_x : A'_x \longrightarrow A_x$$

is an isomorphism and similarly $h_x : S'_x \longrightarrow S_x$ is an isomorphism.

Let

$$h : (A', S') \longrightarrow (A, S)$$

be given by $h|(A'_x, S'_x) = h_x$. If $f \in \Gamma(U, \mathcal{A})$ and f_U is the induced section in $\mathcal{A}'$, given by $\bar{f}(x) = p_{xU}f$, then

$$h\bar{f}(x) = h_x \bar{f}(x) = h_x p_{xU}f = h_{xU}f = f(x)$$

and thus $h(\bar{f}(U)) = f(U)$. The same holds if $s \in \Gamma(U, \mathcal{S})$.

Thus h is an isomorphism of stalks for each x and, since it maps sections f_U onto sections $f(U)$, then h is a local homeomorphism and hence is continuous; hence is a sheaf homomorphism.

Remark 2.18: We identify $(\mathcal{A}', \mathcal{S}')$ with $(\mathcal{A}, \mathcal{S})$ under this isomorphism. For if $a \in \mathcal{A}$, $h^{-1}a$ is the set of all sections $f : U \longrightarrow \mathcal{A}$ where $f(U)$ contains a . Similarly for $h^{-1}s$.

1. Definition of Coherent Sheaves

The property of coherence is such that the theory of coherent sheaves becomes an additive theory and shows that the modules entering in this theory are expected to be of finite type. In applications to algebraic geometry and differential topology, it turns out that most sheaves are coherent.

Remark 3.19: If $\mathcal{A} = (A, \alpha, X)$ is a sheaf of rings with unity, then $\mathcal{A}$ itself is a sheaf of $\mathcal{A}$ -modules and there is, for any index set I , a direct sum $\coprod_{i \in I} \mathcal{A}^i$, where each direct summand is $\mathcal{A}$. This is again a sheaf of $\mathcal{A}$ -modules.

Definition 3.21: Let $\mathcal{A} = (A, \alpha, X)$ be a sheaf of rings with unity and $\mathcal{S} = (S, \sigma, X)$ a sheaf of $\mathcal{A}$ -modules. For each open set $U \subset X$, let

$$f : \coprod_{i \in I} \mathcal{A}^i|_U \longrightarrow \mathcal{S}|_U$$

be a homomorphism. The kernel of f is said to be the sheaf of relations between the sections

$$f_i : U \longrightarrow \mathcal{S}|_U$$

where $f_i = fh^i_1$, i.e.,

$$U \xrightarrow{1} \mathcal{A}|_U \xrightarrow{h^i} \coprod_{i \in I} \mathcal{A}^i|_U \xrightarrow{f} \mathcal{S}|_U. \quad (91)$$

Remark 3.20: If $f_1, \dots, f_k$ are sections over an open set

$U \subset X$ of a sheaf $\mathcal{S} = (S, \sigma, X)$ of $\mathcal{A}$ -modules, we denote by

$$\text{Rel}_x(f_1, \dots, f_k),$$

where $x \in U$, the set of k -tuples $(a^1, \dots, a^k)$ or $\bigsqcup_{i=1}^k a^i$, called a relation, $a^i \in A_x^i$ or $a^i \in \bigsqcup_{i=1}^k A_x^i$ such that

$$a^1 f_1(x) + \dots + a^k f_k(x) = 0$$

or

$$\bigsqcup_{i=1}^k a^i f_i(x) = 0,$$

and by

$$\text{Rel}_U(f_1, \dots, f_k)$$

that subsheaf of

$$\bigsqcup_{i=1}^k A^k|_U$$

whose stalks are the modules $\text{Rel}_x(f_1, \dots, f_k)$.

Proposition 3.19: Let $\mathcal{A} = (A, \alpha, X)$ be a sheaf of rings with unity and $\mathcal{S} = (S, \sigma, X)$ a sheaf of $\mathcal{A}$ -modules. If $\{f_i\}_{i \in I}$ is a system of sections of $\mathcal{S}$ over an open set $U \subset X$, then each f_i defines a homomorphism (again denoted by f_i)

$$f_i : \mathcal{A}|_U \longrightarrow \mathcal{S}|_U$$

where $f_i(a) = a \cdot f_i(x)$, $a \in A_x$. Then the system $\{f_i\}_{i \in I}$ of homomorphisms defines a homomorphism

$$f : \bigsqcup_{i \in I} \mathcal{A}^i|_U \longrightarrow \mathcal{S}|_U.$$

Proof: Since each section f_i is continuous and since $\mathcal{A}$ is also a sheaf of $\mathcal{A}$ -modules, it follows that each f_i is a homomorphism of sheaves. The last part is clearly true.

Proposition 3.20: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules and if

$$f : \bigsqcup_{i=1}^k \mathcal{A}|_U \longrightarrow \mathcal{S}|_U$$

is a homomorphism for each open set $U \subset X$, then the sheaf of relations between the sections

$$f_i : U \longrightarrow \mathcal{S}|_U$$

is a sheaf and this sheaf is contained in

$$\bigsqcup_{i=1}^k \mathcal{A}|_U. \quad (2, (0))$$

Proof: Since

$$\text{Rel}_U(f_1, \dots, f_k) = \ker f$$

and since the kernel of any sheaf homomorphism is a sheaf, then

$$\text{Rel}_U(f_1, \dots, f_k)$$

is a sheaf. Now we show that $\text{Rel}_U(f_1, \dots, f_k)$ is an

$\mathcal{A}|_U$ -module of $\bigsqcup_{i=1}^k \mathcal{A}|_U$. For any $x \in U$ we have that

$$f_x(a^1, \dots, a^k) = a^1(f_1)_x + \dots + a^k(f_k)_x$$

for all $(a^1, \dots, a^k) \in \bigsqcup_{i=1}^k \mathcal{A}_x^i$, and hence

$$(\text{Rel}(f_1, \dots, f_k))_x = \text{Rel}((f_1)_x, \dots, (f_k)_x).$$

The other conditions can easily be verified.

The sheaf of relations between the sections $f_1, \dots, f_k$

is described by the following proposition.

Proposition 3.21: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of $\mathcal{A}$ -modules and if

$$f : \bigsqcup_{i \in I} \mathcal{A}^i|_U \longrightarrow \mathcal{S}|_U$$

is a homomorphism for each open set $U \subset X$, then the element of $(\ker f)_x$ for each $x \in U$ are the elements $\bigsqcup_i a^i$ of $\bigsqcup_i \mathcal{A}_x^i$ for which

$$\bigsqcup_i a^i f_i(x) = 0. \quad (91, (-))$$

Proof: For each element $\bigsqcup_i a^i$, only a finite number of the a^i being different from zero, then

$$\bigsqcup_i a^i = \bigsqcup_{j=1}^p h^{ij} a^{ij}$$

where $h^i : \mathcal{A}|_U \longrightarrow \bigsqcup_i \mathcal{A}^i|_U$ and

$$\begin{aligned} f\left(\bigsqcup_i a^i\right) &= f\left(\bigsqcup_{j=1}^p h^{ij} a^{ij}\right) = \bigsqcup_{j=1}^p f h^{ij}(a^{ij}) \\ &= \bigsqcup_{j=1}^p a^{ij} f h^{ij}(1_x) = \bigsqcup_{j=1}^p a^{ij} f_{i_j}(x) \\ &= \bigsqcup_i a^i f_i(x). \end{aligned}$$

Definition 3.22: Let $\mathcal{S} = (S, \sigma, X)$ be a sheaf of $\mathcal{A}$ -modules. A sheaf $\mathcal{S}$ is said to be of finite type if, for each $x \in X$, there is an open set U , $x \in U$, such that each stalk of $\mathcal{S}|_U$ is generated by the same finite number of sections $f_1, \dots, f_k$ over U . (91)

Proposition 3.22: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of A -modules of finite type and if $f_1, \dots, f_k$ are sections of $\mathcal{S}$ in a neighborhood of $x \in X$ which generate the stalk at x (i.e., whose canonical images $f_1(x), \dots, f_k(x) \in S_x$ generate S_x as an A_x -module), then $f_1, \dots, f_k$ generate the stalk at y for all y sufficiently near x . (50, (+))

Proof: Since $\mathcal{S}$ is a sheaf of finite type, there exist sections $t_1, \dots, t_p$ of $\mathcal{S}$ in a neighborhood of x which generate the stalk at every point of that neighborhood. But since $f_1(x), \dots, f_k(x)$ generate S_x , there exist sections c_{ij} in a neighborhood of x such that

$$t_i(x) = \sum_{j=1}^k c_{ij}(x) \cdot f_j(x)$$

for $i = 1, \dots, p$. But then this equation must hold in a neighborhood of x , and so the $f_j(y)$ generate S_y for all y in this last neighborhood, since the $t_i(y)$ do.

Definition 3.23: A sheaf $\mathcal{S} = (S, \sigma, X)$ of A -modules is said to be a coherent sheaf if:

CS₁ : The sheaf $\mathcal{S}$ is of finite type.

CS₂ : If $f_1, \dots, f_k$ are any finite number of sections of $\mathcal{S}$ over an open set $U \subset X$, then the sheaf of relations between these finite number of sections is of finite type. (91)

Remark 3.21: The definition of coherence is a local property; that is, if every point has a neighborhood U such that $\mathcal{S}|_U$ is coherent, then $\mathcal{S}$ is coherent.

Proposition 3.23: A sheaf $\mathcal{S} = (S, \sigma, X)$ of A -modules is coherent iff every $x \in X$ has a neighborhood in which the induced sheaf is coherent. (2, (-))

Proof: The proof is not given.

Proposition 3.24: If $\mathcal{S} = (S, \sigma, X)$ is a coherent sheaf of A -modules, then $\mathcal{S}$ is locally isomorphic to the cokernel of the homomorphism

$$h : \bigsqcup_{i=1}^p A^p \longrightarrow \bigsqcup_{i=1}^q A^q . \quad (91, (-))$$

Proof: The proof is not given.

2. Homomorphisms, Coherent Subsheaves and Quotient Sheaves

Theorem 3.23: A subsheaf $\mathcal{R} = (R, \rho, X)$ of a coherent sheaf $\mathcal{S} = (S, \sigma, X)$ is coherent iff it is of finite type. (91, (0))

Proof: Since $\mathcal{R}$ is of finite type, we need only verify (CS_2) of the definition of coherence. Any section of $\mathcal{R}$ is a section of $\mathcal{S}$, since R_x is a subgroup of S_x . Hence, the sheaf of relations of any finite number of sections of $\mathcal{R}$ is the sheaf of relations between these sections, considered as sections of $\mathcal{S}$. Since $\mathcal{S}$ is coherent, the sheaf of relations is of finite type.

Theorem 3.24: If $\mathcal{R} = (R, \rho, X)$ and $\mathcal{S} = (S, \sigma, X)$ are coherent sheaves of $\mathcal{A}$ -modules and if $\mathcal{R}$ is a subsheaf of $\mathcal{S}$, then the quotient sheaf $\mathcal{S}/\mathcal{R}$ is coherent. (91, (0))

Proof: For each $x \in X$ there is a neighborhood U in which a finite number of sections $f_1, \dots, f_k$ of $\mathcal{S}|_U$ generate the stalk at every $y \in U$. The images of $f_1, \dots, f_k$ under the natural homomorphism of $\mathcal{S}$ into $\mathcal{S}/\mathcal{R}$ are sections of $\mathcal{S}|_U/\mathcal{R}|_U$ generating the quotient stalk at every $y \in U$. Hence

$$\mathcal{S}/\mathcal{R}$$

is globally finitely generated.

To show that the sheaf of relations of any finite number of sections of $\mathcal{S}/\mathcal{R}$ is locally finitely generated, we consider an open set $V \subset X$, sections $t_1, \dots, t_k$ of $\mathcal{S}|_V/\mathcal{R}|_V$, a point $x \in V$ and a neighborhood U of x in V . We construct sections $f_1, \dots, f_k$ of $\mathcal{S}$ as follows: For each fixed j ,

$$(t_j)_x \in S_x / R_x$$

is the image of an $s_j \in S_x$ under the natural homomorphism

$$h : \mathcal{S} \longrightarrow \mathcal{S}/\mathcal{R}$$

Since $\mathcal{S}$ is coherent, then in a smaller neighborhood of x , there exists a section f_j of $\mathcal{S}$ with

$$(f_j)_x = s_j.$$

Let t'_j be the image of f_j under the homomorphism h . Then

$$(t'_j)_x = (t_j)_x$$

implying that

$$t'_j = t_j$$

near x . Therefore, there is a neighborhood U of x in which we may assume that t_j is the image of f_j , $j = 1, \dots, k$,

and since $\mathcal{R}$ is coherent, then $g_1, \dots, g_p$ are sections of

$\mathcal{R}|U$ generating the stalk at every $y \in U$. Now consider any element of $(\text{Rel}(t_1, \dots, t_k))_x$. Then it is a relation

$$(a^1, \dots, a^k) \in \bigsqcup_{j=1}^k \mathcal{A}_x^j \text{ with}$$

$$\bigsqcup_{j=1}^k a^j(t_j)_x = 0.$$

But $\bigsqcup_{j=1}^k a^j(t_j)_x = 0$ means that

$$\bigsqcup_{j=1}^k a^j(f_j)_x \in \mathcal{R}_x$$

which means

$$\bigsqcup_{j=1}^k a^j(f_j)_x = - \bigsqcup_{i=1}^p b^j(g_i)_x,$$

so that

$$(a^1, \dots, a^k, b^1, \dots, b^p) \in \mathcal{A}_x^{k+p}$$

is a relation between the sections $f_1, \dots, f_k, g_1, \dots, g_p$

of $\mathcal{S}$, a coherent sheaf. Thus in some neighborhood $W \subset U$ of x there are sections

$$c_1^q, \dots, c_k^q, r_1^q, \dots, r_p^q,$$

$q = 1, \dots, n$, of $\text{Rel}(f_1, \dots, f_k, g_1, \dots, g_p)|_W$ over W

such that

$$(a^1, \dots, a^k, b^1, \dots, b^p) = \prod_{q=1}^n e^q(c_1^q, \dots, c_k^q, r_1^q, \dots, r_p^q),$$

$e^q \in A_x^q$. Hence

$$a^i = \prod_{q=1}^n e^q(c_j^q)_x,$$

$j = 1, \dots, k$. Now $(c_1^q, \dots, c_k^q)$, $q = 1, \dots, n$, are sections of $\text{Rel}(t_1, \dots, t_k)|_W$ over W since they are continuous

maps of W into $(a^1, \dots, a^k) \in A_x^k$ such that

$$\prod_{j=1}^k a^j f_j + \prod_{i=1}^p b^i g_i = 0; \text{ i.e., } \prod_{j=1}^k a^j f_j \in R_x; \prod_{j=1}^k a^j t_j = 0.$$

Theorem 3.25: If $\mathcal{S} = (S, \sigma, X)$ is a coherent sheaf of $\mathcal{A}$ -modules and if $f_1, \dots, f_k$ are sections of $\mathcal{S}$, then the sheaf of relations $\text{Rel}(f_1, \dots, f_k)$ is a coherent sheaf. (50, (+))

Proof: Now $\text{Rel}(f_1, \dots, f_k)$ is a subsheaf of $\mathcal{A}^k$, a coherent sheaf, and is of finite type by the definition of coherence of $\mathcal{S}$.

Theorem 3.26: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are coherent sheaves of $\mathcal{A}$ -modules and if

$$h: \mathcal{S} \longrightarrow \mathcal{R}$$

is a homomorphism of $\mathcal{S}$ into $\mathcal{R}$, then $\text{im } h$, $\text{coim } h$, $\text{ker } h$ and $\text{coker } h$ are coherent sheaves. (50, (+))

Proof: To establish the coherence of $\text{im } h = h(\mathcal{S})$ and $\ker h$ we need only show that they are locally finitely generated since we already know that $\text{im } h$ is a subsheaf of $\mathcal{Q}$ and $\ker h$ is a subsheaf of $\mathcal{S}$.

Since $\mathcal{S}$ is of finite type, every $x \in X$ has a neighborhood U in which a finite number of sections $f_1, \dots, f_k$ of $\mathcal{S}|_U$ over U generate $(\mathcal{S}|_U)_y$, $y \in U$. Now their images under h generate $(\text{im } h|_U)_y$, $y \in U$. Thus $\text{im } h$ is of finite type.

For $\ker h$ take x , U and $f_1, \dots, f_k$ as above. Since $h(f_1), \dots, h(f_k)$ are sections of $\mathcal{Q}|_U$ over U , then $\text{Rel}_U(h(f_1), \dots, h(f_k))$ is a coherent sheaf. Consider the following mapping $g : (\text{Rel}_U(h(f_1), \dots, h(f_k)))_y \longrightarrow S_y$, $y \in U$:

$$(a^1, \dots, a^k) \longrightarrow \bigoplus_{i=1}^k a^i (f_i)_y.$$

Since

$$h\left(\bigoplus_{i=1}^k a^i (f_i)_y\right) = \bigoplus_{i=1}^k a^i h(f_i)_y = 0,$$

then g is a homomorphism of $\text{Rel}_U(h(f_1), \dots, h(f_k))$ into $\ker h|_U$. But every element of $(\ker h|_U)_y$ is of the form

$$\bigoplus_{i=1}^k a^i h(f_i)_y \text{ with } a^i \in A_x^1 \text{ and } \bigoplus_{i=1}^k a^i h(f_i)_y = 0. \text{ Hence}$$

g is onto. Therefore $\ker h|_U$ is the image under a homomorphism of a coherent sheaf and thus is coherent by the first part.

Let $\mathcal{A}$ be a $\mathbb{K}$ -algebra. For any $\mathcal{A}$ -module M , we define the annihilator of M as $\text{Ann}_{\mathcal{A}}(M) = \{a \in \mathcal{A} \mid aM = 0\}$. This is a two-sided ideal of $\mathcal{A}$. If M is a faithful $\mathcal{A}$ -module, then $\text{Ann}_{\mathcal{A}}(M) = 0$.

Let $\mathcal{A}$ be a $\mathbb{K}$ -algebra and M an $\mathcal{A}$ -module. We define the annihilator of M as $\text{Ann}_{\mathcal{A}}(M) = \{a \in \mathcal{A} \mid aM = 0\}$. This is a two-sided ideal of $\mathcal{A}$. If M is a faithful $\mathcal{A}$ -module, then $\text{Ann}_{\mathcal{A}}(M) = 0$. We also define the annihilator of an element $m \in M$ as $\text{Ann}_{\mathcal{A}}(m) = \{a \in \mathcal{A} \mid am = 0\}$. This is a left ideal of $\mathcal{A}$. If M is a faithful $\mathcal{A}$ -module, then $\text{Ann}_{\mathcal{A}}(m) = 0$ for all $m \in M$.

Let $\mathcal{A}$ be a $\mathbb{K}$ -algebra and M an $\mathcal{A}$ -module. We define the annihilator of M as $\text{Ann}_{\mathcal{A}}(M) = \{a \in \mathcal{A} \mid aM = 0\}$. This is a two-sided ideal of $\mathcal{A}$. If M is a faithful $\mathcal{A}$ -module, then $\text{Ann}_{\mathcal{A}}(M) = 0$.

$$\text{Ann}_{\mathcal{A}}(M) = \{a \in \mathcal{A} \mid aM = 0\}$$

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3. Exact Sequences of Coherent Sheaves

Theorem 3.27: If $\mathcal{S}' = (S', \sigma', X)$, $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{S}'' = (S'', \sigma'', X)$ are sheaves of $\mathcal{O}$ -modules and if

$$0 \longrightarrow \mathcal{S}' \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\psi} \mathcal{S}'' \longrightarrow 0$$

is an exact sequence of homomorphisms of sheaves and if any two of the sheaves $\mathcal{S}'$, $\mathcal{S}$, $\mathcal{S}''$ are coherent, then the third sheaf is coherent. (91, (0))

Proof: Suppose $\mathcal{S}$ and $\mathcal{S}''$ are coherent. Given $x \in X$ we want to show that $\mathcal{S}'|_U$ is coherent for some neighborhood U of x in X . Since $\mathcal{S}$ is of finite type, there is an epimorphism

$$\tau : \bigoplus_{i=1}^p \mathcal{O}^i|_U \longrightarrow \mathcal{S}|_U$$

where U is some neighborhood of x in X . Let $\rho = (\psi|_U) \cdot \tau$ and $\mathcal{P} = \ker \rho$. Then

$$0 \longrightarrow \mathcal{P} \xrightarrow{\delta} \bigoplus_{i=1}^p \mathcal{O}^i|_U \xrightarrow{\rho} \mathcal{S}''|_U \longrightarrow 0$$

is exact where δ is the canonical injection. Since $\mathcal{S}''$ is coherent, it follows that $\mathcal{P}$ is of finite type. Hence $\tau(\mathcal{P})$ is of finite type. Since $\tau(\mathcal{P})$ is a submodule of $\mathcal{S}|_U$ and $\mathcal{S}$ is coherent, it follows that $\tau(\mathcal{P})$ is coherent. Now $\psi|_U$ maps $\mathcal{S}'|_U$ isomorphically onto $\tau(\mathcal{P})$ and hence $\mathcal{S}'|_U$ is coherent.

Suppose $\mathcal{S}'$ and $\mathcal{S}$ are coherent. Then in particular $\mathcal{S}$ is of finite type and hence so is $\mathcal{S}''$. To show that $\mathcal{S}''$ satisfies

(CS_2) , let sections $h_1, \dots, h_p$ of $\mathcal{S}''$ on an open set V in X be given. We want to show that, given $x \in V$ there exists a neighborhood U of x in V such that $\text{Rel}(h_1|_U, \dots, h_p|_U)$ is a finite $\mathcal{Q}|_U$ -module. Since ψ is an epimorphism, we can find sections $g'_1, \dots, g'_p$ of $\mathcal{S}$ on a neighborhood V' of x in V such that $\psi(g'_i) = h_i|_{V'}$ for $i = 1, \dots, p$. Since $\mathcal{S}'$ is of finite type, we can find sections $f_1, \dots, f_q$ of $\mathcal{S}'$ on a neighborhood U of x in V' which generate $\mathcal{S}'$ on U . Let $g_i = g'_i|_U$. Then $g_1, \dots, g_p, \varphi(f_1), \dots, \varphi(f_q)$ are sections of $\mathcal{S}$ on U and $\psi(g_i) = h_i|_U$. $\text{Rel}(g_1, \dots, g_p, \varphi(f_1), \dots, \varphi(f_q))$ is of finite type because $\mathcal{S}$ is coherent.

Therefore, given $y \in U$ we can find sections

$$(a_j^1, \dots, a_j^p, b_j^1, \dots, b_j^q), \quad j = 1, \dots, J, \quad \text{of } \mathcal{A}^{p+q}$$

on a neighborhood U' of $y \in U$ which generate

$\text{Rel}(g_1, \dots, g_p, \varphi(f_1), \dots, \varphi(f_q))$ on U' . For any $z \in U$

and any $(r^1, \dots, r^p) \in \mathcal{Q}_z^p$ we have:

$$(r^1, \dots, r^p) \in (\text{Rel}(h_1, \dots, h_p))_z$$

$$\iff r^1(g_1)_z + \dots + r^p(g_p)_z \in \text{im } \varphi_z \iff \text{there exist}$$

$$(s^1, \dots, s^q) \in \mathcal{Q}_z^q \text{ such that}$$

$$r^1(g_1)_z + \dots + r^p(g_p)_z + s^1(\varphi(f_1))_z + \dots + s^q(\varphi(f_q))_z = 0 ;$$

that is,

$(r^1, \dots, r^p, s^1, \dots, s^q) \in (\text{Rel}(g_1, \dots, g_p, \varphi(f_1), \dots, \varphi(f_q)))_Z$.

Therefore the sections $(a_j^1, \dots, a_j^p)$, $(j = 1, \dots, J)$, of $\mathcal{Q}^p$ on U' generate $\text{Rel}(h_1, \dots, h_p)$ on U' . This shows that $\text{Rel}(h_1|_U, \dots, h_p|_U)$ is a finite $\mathcal{Q}|_U$ -module.

Suppose $\mathcal{J}'$ and $\mathcal{J}''$ are coherent. Then $\mathcal{J}'$ and $\mathcal{J}''$ are of finite type, and hence given $x' \in X$ we can find sections $u_1, \dots, u_n$ of $\mathcal{J}'$ and sections $v_1, \dots, v_m$ of $\mathcal{J}''$ on a neighborhood Y of x' in X such that these sections generate, respectively, $\mathcal{J}'$ and $\mathcal{J}''$ on Y . Since ψ is an epimorphism, we can find sections $w_1, \dots, w_m$ of $\mathcal{J}$ on a neighborhood Z of x' in Y such that $\psi(w_k) = v_k|_Z$ for $k = 1, \dots, m$.

Since the given sequence is exact, it follows that

$$\varphi(u_1)|_Z, \dots, \varphi(u_n)|_Z, w_1, \dots, w_m$$

generate $\mathcal{J}$ on Z . This shows that $\mathcal{J}$ is of finite type. To prove that $\mathcal{J}$ satisfies condition (CS_2) , let sections

$g_1, \dots, g_p$ of $\mathcal{J}$ on an open set V in X be given. We want

to show that given $x \in V$ there exists a finite number of sections of $\mathcal{Q}^p$ on a neighborhood U of x in V which generate

$\text{Rel}(g_1, \dots, g_p)$ on U . Since $\mathcal{J}''$ is coherent, we can find

sections $(a_e^1, \dots, a_e^p)$, $(e = 1, \dots, d)$, of $\mathcal{Q}^p$ on a

neighborhood V' of x in V which generate $\text{Rel}(\psi(g_1), \dots, \psi(g_p))$

on V' . Let

$$g'_e = \bigsqcup_{i=1}^p a_e^i g_i \in \mathcal{S}|_{V'},$$

$e = 1, \dots, d$. Since the given sequence is exact, there exists a neighborhood V'' of x in V' such that $g'_e|_{V''} \in \varphi(\mathcal{S}'|_{V''})$

for $e = 1, \dots, d$. Since $\mathcal{S}'$ is coherent and φ is a monomorphism, we can find sections $(b_c^1, \dots, b_c^d)$, $(c = 1, \dots, n')$, of $\mathcal{Q}^d$ on a neighborhood U of x in V'' which generate $\text{Rel}(g'_1, \dots, g'_d)$

on U . Let

$$o_c^i = \bigsqcup_{e=1}^d b_c^e a_e^i \in \mathcal{A}|_U,$$

$c = 1, \dots, n'$. We shall show that the sections $(o_c^1, \dots, o_c^p)$, $(c = 1, \dots, n')$, of $\mathcal{A}^p$ on U generate $\text{Rel}(g_1, \dots, g_p)$ on U , and this will complete the proof. So let $y \in U$ be given and let

$$\bar{g}_i = (g_i)_y, \bar{a}_e^i = (a_e^i)_y, \bar{g}'_e = (g'_e)_y, \bar{b}_c^e = (b_c^e)_y, \bar{o}_c^i = (o_c^i)_y.$$

We have to show that the elements $(\bar{o}_c^1, \dots, \bar{o}_c^p)$, $(c = 1, \dots, n')$, in $\mathcal{A}_y^p$ generate $\text{Rel}(\bar{g}_1, \dots, \bar{g}_p)$ as an $\mathcal{A}_y$ -module. First

$$\bigsqcup_{i=1}^p \bar{o}_c^i \bar{g}_i = \bigsqcup_{e=1}^d \bigsqcup_{i=1}^p \bar{b}_c^e \bar{a}_c^i \bar{g}_i = \bigsqcup_{e=1}^d \bar{b}_c^e \bar{g}'_e = 0$$

and hence $(\bar{o}_c^1, \dots, \bar{o}_c^p) \in \text{Rel}(\bar{g}_1, \dots, \bar{g}_p)$ for $c = 1, \dots, n'$.

Next, let $(r^1, \dots, r^p) \in \text{Rel}(\bar{g}_1, \dots, \bar{g}_p)$ be given. Then

$$r^1 \bar{g}_1 + \dots + r^p \bar{g}_p = 0$$

and hence

$$r^1 \psi_y(\bar{g}_1) + \dots + r^p \psi_y(\bar{g}_p) = 0.$$

Therefore we can find $s^1, \dots, s^d$ in A_y such that

$$r^i = \prod_{e=1}^d s_{a_e}^{e-i}$$

for $i = 1, \dots, p$, and then

$$\prod_{e=1}^d s_{g'_e}^{e-i} = \prod_{i=1}^p \prod_{e=1}^d s_{a_e g'_i}^{e-i} = \prod_{i=1}^p r_{g'_i}^{i-i} = 0 ;$$

that is, $(s^1, \dots, s^d) \in \text{Rel}(\bar{g}'_1, \dots, \bar{g}'_d)$. Hence we can find

$t^1, \dots, t^{n'}$ in A_y such that

$$s^e = \prod_{c=1}^{n'} t_{b_c}^{c-e}$$

for $e = 1, \dots, d$, and then for $i = 1, \dots, p$ we get

$$r^i = \prod_{e=1}^d s_{a_e}^{e-i} = \prod_{e=1}^d \left(\prod_{c=1}^{n'} t_{b_c}^{c-e} \right)_{a_e}^{-i} = \prod_{c=1}^{n'} t_{o_c}^{c-i} .$$

Therefore $(r^1, \dots, r^p)$ is in the A_y -module generated by the elements $(\bar{o}_c^1, \dots, \bar{o}_c^p)$, $(c = 1, \dots, n')$.

Theorem 3.28: If $\mathcal{Q} = (A, \alpha, X)$ is a coherent sheaf of rings with unity and $\mathcal{S} = (S, \sigma, X)$ a sheaf of $\mathcal{Q}$ -modules, then $\mathcal{S}$ is coherent iff for each point x there is an open set U with $x \in U$ and an exact sequence

$$\prod_{i=1}^r \mathcal{Q}^i|_U \xrightarrow{g} \prod_{j=1}^s \mathcal{Q}^j|_U \xrightarrow{h} \mathcal{S}|_U \longrightarrow 0 . \quad (91, (0))$$

Proof: To prove the necessity we see that since $\mathcal{S}$ is coherent, property (CS_1) of the Definition 3.23 implies the existence of the homomorphism h and property (CS_2) implies the existence of the homomorphism g .

To prove the sufficiency, we see that since $\mathcal{A}$ is coherent, $\mathcal{A}|_U$ is coherent in U for each open set U and so are

$$\bigsqcup_{i=1}^r \mathcal{A}^i|_U \quad \text{and} \quad \bigsqcup_{j=1}^s \mathcal{A}^j|_U.$$

As the image of $\bigsqcup_{i=1}^r \mathcal{A}^i|_U$, $\text{im } g$ has property (CS_1) , and

as a subsheaf of $\bigsqcup_{j=1}^s \mathcal{A}^j|_U$, $\text{im } g$ has property (CS_2) . Thus

$\text{im } g$ is a coherent sheaf, and so there is an exact sequence:

$$0 \longrightarrow \text{im } g \longrightarrow \bigsqcup_{i=1}^s \mathcal{A}^i|_U \longrightarrow \mathcal{S}|_U \longrightarrow 0.$$

Hence, since two of the sheaves are coherent, then the third $\mathcal{S}|_U$ is coherent for a neighborhood of each point x and hence $\mathcal{S}$ is a coherent sheaf.

4. Operations on Coherent Sheaves

Theorem 3.29: If $\mathcal{S} = (S, \sigma, X)$ and $\mathcal{R} = (R, \rho, X)$ are coherent sheaves of $\mathcal{A}$ -modules, then the direct sum of $\mathcal{S}$ and $\mathcal{R}$ is a coherent sheaf. (50, (+))

Proof: Let $\mathcal{S} \oplus \mathcal{R}$ denote the direct sum of $\mathcal{S}$ and $\mathcal{R}$. It is clear that $\mathcal{S} \oplus \mathcal{R}$ is of finite type. To verify (CS_2) let

$$f_1 + g_1, \dots, f_r + g_r$$

be sections of $\mathcal{S} \oplus \mathcal{R}$ in a neighborhood U of $x \in X$. We wish to show that the sheaf of relations among the $f_i + g_i$ is of finite type. By restricting U if necessary, we may find sections d_j^i of $\mathcal{Q}|U$, $i = 1, \dots, r$, $j = 1, \dots, t$, which generate the sheaf of relations among the f_i , $\text{Rel}_U(f_1, \dots, f_r)$.

Let

$$u_j = \bigoplus_{i=1}^r d_j^i g_i$$

which are then sections of $\mathcal{R}$, and by restricting U again if necessary, find sections h_k^j of $\mathcal{Q}|U$, $j = 1, \dots, t$,

$k = 1, \dots, s$, which generate $\text{Rel}_U(u_1, \dots, u_t)$. Now we

show that the sections

$$\bigoplus_{j=1}^t d_j^i h_k^j \in \mathcal{Q}|U$$

generate $\text{Rel}_U(f_1 + g_1, \dots, f_r + g_r)$. For suppose $y \in U$ and

$(a^1, \dots, a^r)$ are sections of $\mathcal{Q}$ near y such that

$$(1) \quad \bigoplus_{i=1}^r (a^i)_y ((f_i + g_i))_y = 0.$$

Then in particular

$$\bigoplus_{i=1}^r (a^i)_y (f_i)_y = 0.$$

So there exist $(b^1, \dots, b^t) \in \mathcal{Q}$ in a neighborhood of y such that

$$(2) \quad (a^i)_y = \bigsqcup_{j=1}^t (b^j)_y (d_j^i)_y .$$

Now rewriting (1) above we find that

$$\bigsqcup_{i=1}^r \bigsqcup_{j=1}^t (b^j)_y (d_j^i)_y (g_i)_y = 0 ,$$

in other words, that

$$\bigsqcup_{j=1}^t (b^j)_y (u_j)_y = 0 .$$

Hence we can find sections $(c^1, \dots, c^s) \in \mathcal{Q}$ near y such that

$$(b^j)_y = \bigsqcup_{k=1}^s (c^k)_y (h_k^j)_y .$$

Combining this with (2) above, we find

$$\begin{aligned} (a^i)_y &= \bigsqcup_{j=1}^t \bigsqcup_{k=1}^s (c^k)_y (h_k^j)_y (d_j^i)_y \\ &= \bigsqcup_{k=1}^s (c^k)_y \left(\bigsqcup_{j=1}^t d_j^i h_k^j \right)_y , \end{aligned}$$

so that

$$\bigsqcup_{j=1}^t d_j^i h_k^j$$

generate the sheaf of relations among the $f_i + g_i$ as required.

So $\mathcal{S}|_U \oplus \mathcal{R}|_U$ is coherent for a neighborhood of each point x and hence $\mathcal{S} \oplus \mathcal{R}$ is a coherent sheaf.

Theorem 3.30: If $\mathcal{S}$ and $\mathcal{R}$ are two coherent sheaves of $\mathcal{Q}$ -modules, then the tensor product

$$\mathcal{S} \otimes_{\mathcal{Q}} \mathcal{R}$$

is a coherent sheaf. (91, (0))

Proof: By Proposition 3.24, $\mathcal{S}$ is locally isomorphic to the cokernel of the homomorphism

$$h : A^p \longrightarrow A^q .$$

Thus $\mathcal{S} \otimes_a R$ is locally isomorphic to the cokernel of

$$h : A^p \otimes_a R \longrightarrow A^q \otimes_a R .$$

But $A^p \otimes_a R$ and $A^q \otimes_a R$ are respectively isomorphic to R^p and R^q , which are coherent. Hence

$$\mathcal{S} \otimes_a R$$

is coherent by Theorem 3.26.

In this chapter all proofs of the propositions and theorems are omitted. References are understood to be to Serre's fundamental paper (91) .

1. Affine Spaces

Definition 4.24: Let K be a field. An affine space X over the field K is a set of n -tuples $(x_1, \dots, x_n)$ where $x_i \in K$.

Remark 4.22: We want to consider the common solutions of the following polynomial equations

$$f_j(X_i) = 0$$

$i = 1, \dots, n, j = 1, \dots, p$, where $f_j \in K(X_i)$ and K is an algebraically closed field. Let D be the universal domain (i.e., field extension of K such that the degree of transcendency of D/K is infinite and D is algebraically closed). Let $(x) = (x_j)$ denote the common solutions from $D^n = D \times \dots \times D$ (n times). Now we observe that if (x) is a zero for $f_1, \dots, f_p$ then (x) is also a zero for

$$g_1 f_1 + \dots + g_p f_p$$

where g_j are any polynomials of $K(X_i)$. Thus (x) is a zero of the ideal generated by f_j . Instead of considering the original problem we consider the following: Given an ideal $I \subset K(X_i)$ we consider the zeros (x) of I , where (x) is the zero of I for

every $f(X_i) \in I$, we have $f(x) = 0$.

This problem is actually equivalent to our original problem since by a theorem (namely: If K is a field, then $K(X_i)$ is a Noetherian ring), any ideal I of $K(X_i)$ is finitely generated. So to find a zero of I , it suffices to find a zero of only a finite number of elements of I .

Definition 4.25: Let I be an ideal of $K(X)$, where $X = X_1, \dots, X_n$.

The set of zeros of the ideal I is said to be an algebraic set over the field K .

Remark 4.23: We note that the algebraic set can be the empty set. This will occur when $I = K(X)$, since then the identity element $1 \in I$. Also, the algebraic set equals D^n when $I = (0)$, the zero ideal. It is also possible for different ideals to define the same algebraic set.

The following proposition shows that the set of the algebraic sets is closed under union and intersection.

Proposition 4.25: If I, I' are ideals contained in $K(X)$ and if I, I' define algebraic sets V, V' , then the union and intersection are algebraic sets.

Definition 4.26: The topology in an affine space $X = K^p$ of dimension p over the field K defined by the closed sets consisting of algebraic sets in K^p is said to be the Zariski topology.

Remark 4.24: That the so-called Zariski topology is a topology follows from Proposition 4.25.

Proposition 4.26: If X is an affine space over the field K provided with Zariski topology, then a subset of X is closed if it is a set of zeros common to a collection of polynomials $P_i \in K(X_1, \dots, X_p)$, $i = 1, \dots, q$.

Definition 4.27: A topological space X is said to be irreducible if it is not a union of two closed disjoint subsets of X .

Proposition 4.27: If X is an affine space with Zariski topology, then X is an irreducible space.

2. Sheaf of Local Rings

The sheaf of local rings over an affine space is fundamental to the algebraic sheaf theory.

Definition 4.28: A subring of the closed field K^p consisting of rational fractions of the form

$$P(x)/Q(x)$$

where P, Q are polynomials, $Q(x) \neq 0$, is said to be a local ring.

Definition 1.1. The *fundamental group* of a space X is the group of homotopy classes of loops in X based at a point x_0 . It is denoted by $\pi_1(X, x_0)$.

Example 1.1. The fundamental group of a circle is isomorphic to the integers $\mathbb{Z}$.

Proposition 1.1. If X is a simply connected space, then the fundamental group of X is trivial. Conversely, if the fundamental group of X is trivial, then X is simply connected.

Definition 1.2. A space X is called *simply connected* if it is path connected and its fundamental group is trivial.

Example 1.2. A disk is simply connected, but an annulus is not.

2. The fundamental group

The fundamental group of a space X is a group. The identity element is the class of the constant loop.

Definition 1.3. A space X is called *locally simply connected* if every point in X has a simply connected neighborhood.

Example 1.3. A disk is locally simply connected, but an annulus is not.

Remark 4.25: The definition of a local ring may be stated as a ring with unity having a unique maximal ideal. If $x = (x_1, \dots, x_p)$ is an element of $X = K^p$, we denote by O_x the local ring at x .

Definition 4.29: Let X be an affine space over an infinite field K provided with Zariski topology. The map

$$x \longrightarrow P(x)/Q(x), \quad Q(x) \neq 0$$

for each $x \in X$ with values in K is said to be a regular rational map.

Proposition 4.28: If X is an affine space over an infinite field K provided with Zariski topology, then at each $x \in X$ the map $x \longrightarrow P(x)/Q(x), \quad Q(x) \neq 0$, is continuous with values in K .

Theorem 4.31: If $\mathcal{S} = (S, \sigma, X)$ is a sheaf of germs of maps over an affine space X with values in an infinite field K , then the set of local rings O_x for each $x \in X$ forms a subsheaf $\mathcal{O} = (O, \varphi, X)$ of the sheaf $\mathcal{S}$.

Corollary 4.31: The sheaf of local rings $\mathcal{O} = (O, \varphi, X)$ over an affine space with values in a field K is a coherent sheaf of rings.

Definition 4.30: A subset Y of an affine space X is said to be locally closed in X if it is the intersection of an open and closed subset of X .

We extend the above notions to locally closed subspaces of an affine space X .

Proposition 4.29: If Y is a locally closed subspace of an affine space X and if $\mathcal{S}|_Y$ is the sheaf of germs of maps on Y with values in an infinite field provided with Zariski topology, then the operation of restriction of a map defines a canonical homomorphism

$$f : (\mathcal{S}|_X)_x \longrightarrow (\mathcal{S}|_Y)_x .$$

Corollary 4.29: The image of the local ring O_x by f is a subring of $(\mathcal{S}|_Y)_x$.

Proposition 4.30: If X is an affine space and $Y \subset X$ is a locally closed subspace, then the local rings $O_x|_Y$ for each $x \in Y$ form a subsheaf $\mathcal{O} = (O, \varphi, Y)$ of $\mathcal{S}|_Y$.

Remark 4.26: The sheaf in Proposition 4.30 is called the sheaf of local rings over a locally closed affine space Y and is denoted by $\mathcal{O}|_Y$.

Defintion 4.31: Let $\mathcal{O}|_Y$ be a sheaf of local rings over a locally closed affine space Y . A section of $\mathcal{O}|_Y$ over an open set $V \subset Y$.

$$f : V \longrightarrow K$$

where K is an infinite field is said to be a regular map over V .

Theorem 4.32: If $\mathcal{O}|_Y$ is a sheaf of local rings over a locally closed affine space Y , then a section of $\mathcal{O}|_Y$ over an open set $V \subset Y$ is a map

$$f : V \longrightarrow K .$$

where K is an infinite field, which is equal, in the neighborhood of each $x \in V$, to the restriction to V of a regular rational map at x .

Corollary 4.32: The map $f : V \longrightarrow K$ is continuous if V is provided with the induced topology of Y and K with the Zariski topology.

Definition 4.32: Let $U \subset X$, $V \subset Y$ be locally closed subspaces of affine spaces $X = K^p$, $Y = K^q$, p, q integers. A map

$$f : U \longrightarrow V$$

is said to be a regular map over U if

R_1 : f is a continuous map.

R_2 : If $x \in U$ and if $g \in \mathcal{O}_{f(x)}|_V$, then $g \cdot f \in \mathcal{O}_x|_U$.

Remark 4.27: We denote the co-ordinates of the point $f(x)$ by $f_i(x)$, $1 \leq i \leq q$.

Theorem 4.33: If $U \subset X$, $V \subset Y$ are locally closed subspaces of affine spaces X, Y , then a map

$$f : U \longrightarrow V$$

is regular iff

$$f_i : U \longrightarrow K$$

are regular on U for $1 \leq i \leq q$.

Corollary 4.33: The composition of two regular maps is a regular map.

3. Structure of Algebraic Variety

A variety is a (point) set with a structure given by a sheaf of local rings, or given by embedding it in some space. If a variety V is considered as a (point) set, we take into account all points on V whose co-ordinates are in an algebraically closed commutative field K of any characteristic. A topology on V is given by the Zariski topology.

Definition 4.33: An algebraic variety V in an affine space X over a field K (algebraically closed field of any characteristic) is a set provided with two structures:

1. Topological structure

The set V has the Zariski topology.

2. Algebraic structure

The sheaf of germs of maps $\mathcal{O}|_V$ over V with values in K defines a subsheaf $\mathcal{O}|_V$ of local rings.

which are subject to the following axioms:

AV_1 : There exists a finite covering $\mathcal{V} = \{V_i\}_{i \in I}$ of the space V such that each V_i , provided with the induced structure of V , is isomorphic to a locally closed subspace U_i of the affine space X , provided

with the sheaf $\mathcal{O}|_{U_i}$ of local rings $\mathcal{O}_i$.

AV_2 : The diagonal of the product variety $V \times V$ is closed in $V \times V$.

Theorem 4.34: If V is an algebraic variety in an affine space over some algebraically closed field K , then the sheaf

$$\mathcal{O} = \mathcal{O}|_V$$

of local rings is a coherent sheaf of rings over V .

Remark 4.28: The sheaf $\mathcal{O} = \mathcal{O}|_V$ of local rings over an algebraic variety V is fundamental to algebraic sheaf theory. The stalks $\mathcal{O}_x$ are the local rings of V at the various points $x \in V$.

Local sections of $\mathcal{O}$ are then described by locally regular maps f which are meromorphic on V ; and any such map is an element of the function field K of V . The points $x \in V$ where f is not regular (i.e., the points x such that $f \notin \mathcal{O}_x$) form an algebraic subvariety W of V , and thus f defines a section of $\mathcal{O}$ over $U = V - W$ (U is open in the Zariski topology).

If V is an algebraic set, $\mathcal{O}|_V$ can be given locally by embedding parts of V in affine spaces. The sheaves of local rings thus obtained can be pasted together and this is done canonically, as $\mathcal{O}|_V$ is a subsheaf of the germs of numerical maps on V . If there are multiple components this method of pasting together the sheaves on the affine parts is not canonical; in that case the local structure of $\mathcal{O}|_V$ is not sufficient to determine $\mathcal{O}|_V$.

4. Algebraic Sheaves

The purely algebraic theory of sheaves developed by Serre (91) represents the first systematic application of homological algebra to abstract algebraic geometry. The ultimate goal of the cohomological theory of sheaves lies in the direction of projective varieties or even abstract varieties. A preliminary study of sheaves on affine varieties is essential since the set of open coverings of a projective variety V by affine varieties is cofinal with the set of all open coverings of V .

Definition 4.34: A sheaf $\mathcal{S} = (S, \sigma, V)$ over an algebraic variety V is said to be an algebraic sheaf if it is a sheaf of $\mathcal{O}$ -modules, where $\mathcal{O}$ is the sheaf of local rings over V .

Remark 4.29: The sheaf $\mathcal{O} = (\mathcal{O}, \varphi, V)$ of local rings over an algebraic variety V used to define an algebraic sheaf $\mathcal{S} = (S, \sigma, V)$ is called the sheaf of operators. Each stalk $\mathcal{O}_x$ is a local ring and S_x is a module over $\mathcal{O}_x$. The ring structure of $\mathcal{O}_x$ and the module structure of S_x both vary continuously with x .

Proposition 4.31: If $\mathcal{O} = (\mathcal{O}, \varphi, V)$ is a sheaf of local rings over an algebraic variety V , then $\mathcal{O}$ is an algebraic sheaf.

Definition 4.35: Let $\mathcal{S} = (S, \sigma, V)$ and $\mathcal{R} = (R, \rho, V)$ be algebraic sheaves over an algebraic variety V . A homomorphism between algebraic sheaves

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

is said to be algebraic if it is an $\mathcal{U}$ -homomorphism; or equivalently, if for each h_x ,

$$h_x : S_x \longrightarrow R_x,$$

is $\mathcal{O}_x$ -linear; i.e., if for every $x \in V$ the local homomorphism h_x is $\mathcal{O}_x$ -linear.

Remark 4.30: We can define substructures, quotient structures and induced structures; operations of direct sum and tensor product; and notion of exact sequence and coherence of algebraic sheaves.

Theorem 4.35: If h is an algebraic homomorphism of a coherent algebraic sheaf $\mathcal{S}$ onto a coherent algebraic sheaf $\mathcal{R}$ on an algebraic variety V , then h maps $\Gamma(V, \mathcal{S})$ onto $\Gamma(V, \mathcal{R})$.

Theorem 4.36: If $\mathcal{S}$ is a coherent algebraic sheaf on an algebraic variety V , then there is an exact sequence

$$\mathcal{G}^p \longrightarrow \mathcal{G}^q \longrightarrow \mathcal{S} \longrightarrow 0.$$

5. Structure of Analytic Variety

We state the definition of a (complex) analytic variety in terms of sheaves.

Definition 4.36: A (complex) analytic variety V in a complex numerical space C^n of (complex) dimension n is a set provided with two structures:

1. Topological structure

The set V has a (separable) topology.

2. Algebraic structure

The sheaf of germs of continuous (complex) maps defines a subsheaf $\overline{\mathcal{G}}|_V$ of germs of holomorphic maps.

which are subject to the following axiom:

AV_1 : For each $x \in V$, there exists an open set U containing x and sections f_i , $i = 1, \dots, n$, of $\overline{\mathcal{G}}|_V$ over U , zero at points x , such that

(a) f_i define a homeomorphism of U onto an open set of the complex numerical space C^n .

(b) Elements of $\overline{\mathcal{O}}_x$ are the composite functions $f(f_1, \dots, f_n)$, where f is holomorphic at the origin (in C^n).

Remark 4.31: The system of n sections f_i having these properties are called the system of local co-ordinates at the point x . The sheaf $\overline{\mathcal{G}}|_V$ is called the sheaf of germs of holomorphic maps.

Proposition 4.32: If $\bar{\mathcal{O}}$ is a sheaf of germs of holomorphic maps over an analytic variety V , then sections of $\bar{\mathcal{O}}$ over an open set $U \subset V$ are holomorphic maps o into U .

Proposition 4.33: If V is an analytic variety, then the set of all germs of holomorphic maps over V forms a sheaf of rings.

Remark 4.32: Let V be an analytic variety and let $\mathcal{O}_x$, the set of germs of holomorphic maps. Let $x \in V$ and consider holomorphic maps at x , each defined in some neighborhood of x . We say that two such maps are equivalent if they coincide on some neighborhood sufficiently small of x . This is an equivalence relation. The set of all holomorphic maps modulo this equivalence relation is called the set of germs of holomorphic maps at x , denoted by $\bar{\mathcal{O}}_x$. The set of all germs form a ring over each point x .

We introduce a topology in the set of all germs

$\mathcal{O} = \bigcup_{x \in V} \mathcal{O}_x$ as follows: Take any element $f \in \mathcal{O}$. Then $f \in \mathcal{O}_{x_0}$

and is an equivalence class of maps defined in a neighborhood of $x_0 \in V$. Take a representative $g \in \mathcal{O}_x$. Now g is defined in

U_{x_0} , a neighborhood of $x_0 \in V$. Then for each $y \in U_{x_0}$

assign that set in $\mathcal{O}_y$ containing the direct analytic continuation of g , say $\{g_y\}$. Then $\bigcup_{y \in U_{x_0}} \{g_y\}$ is by definition an open

set which form a base for the topology of $\mathcal{O}$. This sheaf is called the sheaf of germs of holomorphic maps.

6. Analytic Sheaves

Definition 4.37: A sheaf $\mathcal{S} = (S, \sigma, V)$ over a (complex) analytic variety V is said to be an algebraic sheaf if it is a sheaf of $\overline{\mathcal{O}}$ -modules, where $\overline{\mathcal{O}}$ is the sheaf of germs of holomorphic maps over V .

Proposition 4.34: If $\overline{\mathcal{O}} = (\mathcal{O}, \varphi, V)$ is a sheaf of germs of holomorphic maps over an analytic variety V , then $\overline{\mathcal{O}}$ is an analytic sheaf.

Definition 4.38: Let $\mathcal{S} = (S, \sigma, V)$ and $\mathcal{R} = (R, \rho, V)$ be analytic sheaves over an analytic variety V . A homomorphism of analytic sheaves

$$h : \mathcal{S} \longrightarrow \mathcal{R}$$

is said to be analytic if it is an $\overline{\mathcal{O}}$ -homomorphism; or equivalently, if for each h_x ,

$$h_x : S_x \longrightarrow R_x,$$

is $\overline{\mathcal{O}}_x$ -linear.

Remark 4.33: We can define substructures, quotient structures and induced structures; operations of direct sum and tensor product, and the notion of exact sequence and coherence of analytic sheaves.

7. Isomorphism of Algebraic and Analytic Sheaves

The study of an algebraic variety V defined over the field of complex numbers can be undertaken from two points of view: the algebraic point of view, in which we are concerned with the local rings of the points of V and with the regular maps (or rational maps) of V into some other variety; and the analytic point of view, in which the notion of holomorphic map on V plays the principal role.

In other words, we have two types of sheaves associated with an algebraic variety V , namely the analytic sheaf $\bar{\mathcal{O}}$ and the algebraic sheaf $\mathcal{O}$. They differ in two important respects: in the topology of their base space V and in the fact that $\bar{\mathcal{O}}_x$ contains $\mathcal{O}_x$ but is a much larger ring than $\mathcal{O}_x$. Therefore in the classical case we have a priori two sheaf theories, or two cohomology theories of V : one is analytic, the other is algebraic. Serre (91) established an "isomorphism" between these two theories.

Every (affine) algebraic variety V over the field of complex numbers can be provided, in a canonical way, with a structure of analytic space. Every coherent algebraic sheaf on V determines a coherent analytic sheaf. When V is a (projective) algebraic variety, we can show, conversely, that every coherent analytic sheaf on V can be obtained also, and in a unique way.

Since the Zariski topology is weaker than the Hausdorff topology, we can define a mapping from the algebraic cohomology groups into the corresponding analytic cohomology groups which commute with homomorphisms in both theories. Now, since the mapping is an isomorphism for zero dimensional cohomology groups and since the sheaves $\mathcal{O}(n)$ ($\mathcal{O}(n)$ is the sheaf which associates another sheaf with any integer n .) have the property that

$$H^q(V, \mathcal{O}) = 0$$

if $q > 0$ for coherent sheaves $\mathcal{F}$ in both theories, then the mapping gives an "isomorphism" of the two sheaf theories in the classical sense.

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